Homological finite derivation type

Juan M. Alonso and Susan M. Hermiller*

Abstract: In 1987 Squier defined the notion of finite derivation type for a finitely presented monoid. To do this, he associated a 2-complex to the presentation. The monoid then has finite derivation type if, modulo the action of the free monoid ring, the 1-dimensional homotopy of this complex is finitely generated. Cremanns and Otto showed that finite derivation type implies the homological finiteness condition left FP_3, and when the monoid is a group, these two properties are equivalent. In this paper we define a new version of finite derivation type, based on homological information, together with an extension of this finite derivation type to higher dimensions, and show connections to homological type FP_n for both monoids and groups.

1. Introduction

In [11], Squier defined a complex associated to a finite presentation of a monoid or group, along with a combinatorial property of this complex known as finite derivation type. His original motivation was to capture much of the information of a finite complete rewriting system for a monoid in a property which is independent of presentation. More recently, Cremanns and Otto [4], Lafont [8], and Pride [9] have independently shown that the finite derivation type property also implies the homological finiteness conditions left and right FP_3 for monoids, and Cremanns and Otto [5] have shown that finite derivation type is equivalent to the property left (and hence right) FP_3 for groups (see also [10] for an alternative proof of this result). For monoids, these conditions are not equivalent. In his original paper, Squier [11] gave an example of a monoid with type left FP_3 which does not have finite derivation type, and more recently Kobayashi and Otto [7] have constructed a monoid which is both left and right FP_3 (and moreover both left and right FP_∞) but which does not have finite derivation type.

For a finitely presented group, type FP_3 is a property of the 2-dimensional homology of the Cayley complex associated to the presentation, implying finite generation as a left module over the integral group ring. A finitely presented monoid also has finite derivation type essentially if the 1-dimensional homotopy of the corresponding Squier complex is finitely generated, modulo an action by the free monoid on the generators. Thus the theorem of Cremanns and Otto shows that the property FP_3 for a group can be reduced in dimension to a property of the 1-dimensional homotopy of another complex. It is natural to ask if this process can be repeated in higher dimensions. In [6], Kobayashi has introduced a property known as a homotopy reduction system, which is similar to finite derivation type in one dimension higher, and has shown that this property implies the homological finiteness condition right FP_4 for finitely presented monoids.

In [12], X. Wang and Pride introduce the notion of finite homological type (in more recent work this has also been referred to as finite homotopy type), which is a finiteness condition on the homology rather than the homotopy of the Squier complex. They also show that for groups, this property is equivalent to the condition FP_3, and for monoids, it implies left and right FP_3. The

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monoid constructed in [7] is also shown there not to have finite homological type, so as above this condition is not equivalent to the property of left and right FP_3 for monoids.

In this paper, we introduce a new definition of homological finite derivation type in all dimensions, in Sec. 3. This definition starts from information about a partial free resolution of the integers over the integral monoid or group ring, and imitates Squier's construction. Since we start with a resolution rather than a finite presentation for the monoid or group, this also allows the monoid to be infinitely presented. Associated to this resolution we introduce a sequence of graphs, one for each dimension n, which capture n-dimensional homological information about the monoid. A monoid then has n-dimensional homological finite derivation type if each of the graphs up to dimension n satisfies a property analogous to Squier's finite derivation type.

In Sec. 4 we study a bimodule structure on a set of pairs of paths in the graphs defined in Sec. 3, and show that the bimodule is isomorphic to the kernel of the corresponding boundary map in the resolution.

In Sec. 5 we use the results in Sec. 4 to prove the main theorem of this paper. This theorem states that for groups, the property of homological finite derivation type in dimension n ($HFDT_n$) is equivalent to the property FP_n, and for monoids, $HFDT_n$ is equivalent to the existence of a length n partial resolution of the integers by finite rank free left, right, or bi-modules over the integral monoid ring (the property left, right, or bi- FP_n, respectively), depending on which type of modules occur in the original resolution to which the graphs are associated.

We begin in Sec. 2 with a discussion of homological finiteness conditions, including the connections between left, right, and bi- FP_n for groups and monoids. We prove that a monoid that has both type left FP_n and right FP_n must also have type bi- FP_n, and the converse is also true for groups. Therefore the results listed above show that finite derivation type and finite homological type each imply the property $HFDT_3$, and the converse is true for groups but not true for monoids. In particular, the monoid example in [7] has type left, right, and bi- FP_3, and hence $HFDT_3$ on the corresponding sides, but does not have finite derivation type nor finite homological type. Section 2 also includes background on Squier's finite derivation type.

2. Background

2.1. Homological finiteness conditions

A group G has *type* FP_n if there is an exact sequence (or partial resolution of $\mathbf{Z}$) $P_n \to \cdots \to P_0 \to \mathbf{Z} \to 0$ with finitely generated free left $\mathbf{Z}G$-modules P_i, and G has type FP_∞ if it has type FP_n for every natural number n.

A monoid M has type *left* FP_n if there is a partial resolution of the integers by finitely generated free left $\mathbf{Z}M$-modules of length n. Similarly M has type *right* FP_n if there is a length n resolution of $\mathbf{Z}$ by finite rank free right $\mathbf{Z}M$-modules and M has type *bi-* FP_n if there is a finite rank free length n resolution of $\mathbf{Z}$ by $(\mathbf{Z}M, \mathbf{Z}M)$-bimodules. The monoid has type left, right, or bi- FP_∞ if it has type left, right, or bi- FP_n for all n, respectively.

For a group G, if P is a left $\mathbf{Z}G$-module, then there is an associated right $\mathbf{Z}G$-module P'. As an abelian group, P' is isomorphic to P with an isomorphism $\phi : P \to P'$, and the right action of G on P' is given by $p \cdot g := \phi(g^{-1} \cdot \phi^{-1}(p))$, where $p \in P'$ and $g \in G$. If P is free, then P' is also free with the same basis. Thus any partial resolution of $\mathbf{Z}$ by finitely generated free left $\mathbf{Z}G$-modules has an associated resolution by finitely generated free right $\mathbf{Z}G$-modules. Similarly, any partial resolution by right modules has an associated left module resolution. Therefore for groups, the properties of left FP_n and right FP_n are equivalent.

Also for a group G, if P is a free $(\mathbf{Z}G, \mathbf{Z}G)$-bimodule, then there is an associated free left $\mathbf{Z}(G \times G)$-module P'' with the same basis, defined via an abelian group isomorphism $\theta : P \to P''$, and action $(g, h) \cdot p := \theta(g \cdot \theta^{-1}(p) \cdot h^{-1})$ for $g, h \in G$ and $p \in P''$. Then any partial resolution of $\mathbf{Z}$ by finitely generated free $(\mathbf{Z}G, \mathbf{Z}G)$-bimodules has an associated resolution by finitely generated free left $\mathbf{Z}(G \times G)$-modules, and the converse is also true. Therefore G has type bi- FP_n iff $G \times G$ has type FP_n. [2, Proposition V.1.1] shows that if G has type FP_n, then so does $G \times G$. Since the group G is a retract of $G \times G$, [1, Theorem 8] shows that if $G \times G$ has type FP_n, then so does G. This proves the following.

Proposition 2.1. *For any group G, the finiteness conditions left FP_n, right FP_n, and bi- FP_n are equivalent.* □

Thus for groups, the side is not mentioned in the FP_n property.

In the case of monoids, however, Daniel Cohen [3] has shown that these properties are not all equivalent; in particular, his paper shows that there is a monoid which has type right FP_∞ but which is not left FP_1. A revision of the discussion above leads to the following connection between these finiteness conditions for monoids.

Proposition 2.2. *If a monoid M satisfies both of the finiteness conditions left FP_n and right FP_n, then M also has type bi- FP_n.*

Proof. Suppose M has type left and right FP_n. Let $L_n \to \cdots \to L_0 \to \mathbf{Z} \to 0$ be a finite rank free partial resolution of $\mathbf{Z}$ by left $\mathbf{Z}M$-modules, and let $R_n \to \cdots \to R_0 \to \mathbf{Z} \to 0$ be a finite rank free partial resolution of $\mathbf{Z}$ by right $\mathbf{Z}M$-modules. Then each abelian group $L_p \otimes_{\mathbf{Z}} R_q$ is a free finite rank $(\mathbf{Z}M, \mathbf{Z}M)$-bimodule. The complexes $L_n \to \cdots \to L_0 \to 0$ and $R_n \to \cdots \to R_0 \to 0$ each have trivial homology groups in dimension greater than 0, and homology group H_0 equal to $\mathbf{Z}$. Define the complex $C_n \to \cdots \to C_0$ to be the tensor product over $\mathbf{Z}$ of these two complexes. That is,

$$C_i := \bigoplus_{p+q=i} L_p \otimes_{\mathbf{Z}} R_q$$

and $\partial_i(l \otimes r) := \partial_p(l) \otimes r + (-1)^p l \otimes \partial_q(r)$ for $l \in L_p$ and $r \in R_q$. Then each C_i is also a free finite rank $(\mathbf{Z}M, \mathbf{Z}M)$-bimodule. The Künneth formula for a tensor product of complexes ([2, Proposition I.0.8]) then applies to show that the complex $C_n \to \cdots \to C_0 \to 0$ has homology groups which are also trivial, except for $H_0(C) = H_0(L) \otimes H_0(R) = \mathbf{Z} \otimes \mathbf{Z} = \mathbf{Z}$. Then the augmented complex $C_n \to \cdots \to C_0 \to \mathbf{Z} \to 0$ is a free finite rank partial resolution of $\mathbf{Z}$ by $(\mathbf{Z}M, \mathbf{Z}M)$-bimodules. Therefore M has type bi- FP_n. □

For a group G, the FP_n property has a connection to topology as well. A $K(G, 1)$-complex is a connected CW complex Y with fundamental group $\pi_1(Y) = G$ and contractible universal cover $\widetilde{Y}$. The cellular chain complex $C_*(\widetilde{Y})$, with the augmentation map to the integers, gives a resolution of $\mathbf{Z}$ by free left $\mathbf{Z}G$-modules. If the group G has a $K(G, 1)$-complex with only finitely many cells in dimension less than or equal to n (and arbitrarily many cells of higher dimension), then the group also has type FP_n.

For any monoid or group M and any integer $n \geq 0$, the property (left, right, or bi-) FP_n is equivalent to the property that for every partial finitely generated free (or projective) resolution

$F_k \to \cdots \to F_0 \to \mathbf{Z} \to 0$ of $\mathbf{Z}M$-modules on the corresponding side with $k < n$, $\ker\{F_k \to F_{k-1}\}$ is finitely generated (see, for example, [2, Theorem 4.3 of Chap. 8]).

For proofs and more detailed information on homological finiteness conditions, we refer the reader to [2].

2.2. Finite derivation type

In this section we give the definition of the graph and homotopy relations associated to a finite monoid presentation, defined by Squier in [11]. Let $\mathcal{P} = \langle A \mid R \rangle$ be a presentation of a monoid M, and let A^* be the free monoid on A.

Definition (Associated graph X). ([11]) This is the graph whose vertices and edges are given by:

(1) Vertices: $V(X) := A^*$.

(2) Edges: $E(X) := \{(a, [\pi_1, \pi_{-1}], b, \epsilon) \mid a, b \in A^*,\ [\pi_1, \pi_{-1}] \in R,\ \epsilon \in \{1, -1\}\ \}$.

(3) $\iota, \tau : E(X) \to V(X)$ are defined by:

$$\iota(e) := a \cdot \pi_\epsilon \cdot b \qquad\qquad \tau(e) := a \cdot \pi_{-\epsilon} \cdot b,$$

where e denotes $(a, [\pi_1, \pi_{-1}], b, \epsilon)$ and $\cdot$ denotes concatenation in A^*.

(4) $()^{-1} : E(X) \to E(X)$ is given by

$$(a, [\pi_1, \pi_{-1}], b, \epsilon)^{-1} := (a, [\pi_1, \pi_{-1}], b, -\epsilon).$$

Next define the set of paths

$$P := \{(e_1, ..., e_m) \mid e_j \in E(X),\ \tau(e_j) = \iota(e_{j+1}) \text{ for each } j\}.$$

Denoting concatenation of paths by $\circ$, we will write $(e_1, ..., e_m)$ as $e_1 \circ \cdots \circ e_m$. For $x \in V(X)$, let (x) denote the constant path at x. Again we have maps $\iota, \tau : P \to V(X)$ defined by

$$\iota(e_1 \circ \cdots \circ e_m) = \iota(e_1) \quad \text{and} \quad \tau(e_1 \circ \cdots \circ e_m) = \tau(e_m).$$

When $M = G$ is a group, the edges and paths in the associated graph X also have the following topological meaning. Suppose Y is the standard complex associated to the presentation $\mathcal{P}$ (in this case considered as a group presentation) of G. An element of $E(X)$ corresponds to a single 2-cell in the universal cover $\widetilde{Y}$, with top π_ϵ and bottom $\pi_{-\epsilon}$, together with a 1-dimensional tail a on the left, and another tail b on the right. An element of P corresponds to a 2-disk, with top $\iota(e_1)$ and bottom $\tau(e_m)$; the interior of the disk consists of a layering of the 2-cells from $e_1, ..., e_m$ in order from top to bottom, with the 2-cells offset from one another horizontally using the tails.

Definition (Action of A^* on P). Given $\alpha \in A^*$ and $e = (a, [\pi_1, \pi_{-1}], b, \epsilon) \in E(X)$, set

$$\alpha \cdot e := (\alpha \cdot a, [\pi_1, \pi_{-1}], b, \epsilon) \quad \text{and} \quad e \cdot \alpha := (a, [\pi_1, \pi_{-1}], b \cdot \alpha, \epsilon),$$

which are edges in $E(X)$. Given a path $p = e_1 \circ \cdots \circ e_k \in P$, set

$$\alpha \cdot p := (\alpha \cdot e_1) \circ \cdots \circ (\alpha \cdot e_m) \quad \text{and} \quad p \cdot \alpha := (e_1 \cdot \alpha) \circ \cdots \circ (e_m \cdot \alpha),$$

which are paths in P.

Definition ($P^{(2)}(X)$).

$$P^{(2)}(X) := \{(p,q) \mid p,q \in P,\ \iota(p) = \iota(q),\ \tau(p) = \tau(q)\}.$$

Definition (D, I).

$$D := \{((e_1 \cdot \iota(e_2)) \circ (\tau(e_1) \cdot e_2), (\iota(e_1) \cdot e_2) \circ (e_1 \cdot \tau(e_2))) \mid e_1, e_2 \in E(X)\}$$
$$I := \{(e \circ e^{-1}, (\iota(e))) \in P^{(2)}(X) \mid e \in E(X)\ \}.$$

Definition (Homotopy relation). A homotopy relation on P is an equivalence relation $\simeq \subseteq P^{(2)}(X)$ such that

(1) $D \cup I \subseteq \simeq$.
(2) If $p, q \in P$, $p \simeq q$, and $\alpha \in A^*$, then $\alpha \cdot p \simeq \alpha \cdot q$ and $p \cdot \alpha \simeq q \cdot \alpha$.
(3) If $p, q, r, s \in P$, $\tau(r) = \iota(p)$, $\iota(s) = \tau(p)$, and $p \simeq q$, then $r \circ p \simeq r \circ q$ and $p \circ s \simeq q \circ s$.

For any set $B \subseteq P^{(2)}(X)$, the smallest possible homotopy relation containing B will be called the homotopy relation generated by B.

Definition (Finite derivation type). The monoid M has finite derivation type, or type FDT, if there is a finite set $B \subseteq P^{(2)}(X)$ for which the homotopy relation generated by B is all of $P^{(2)}(X)$.

If a monoid M has a finite presentation $\mathcal{P} = \langle A \mid R \rangle$, there is an exact sequence of free left $\mathbf{Z}M$-modules

$$F_2 \to F_1 \to F_0 \to \mathbf{Z} \to 0,$$

where each F_i has a basis $\underline{\beta}^i$ with $\underline{\beta}^0 = \{\sigma_1\}$, $\underline{\beta}^1 = A$, $\underline{\beta}^2 = R$, and

$$F_i = \bigoplus_{\sigma \in \underline{\beta}^i} \mathbf{Z}M\sigma.$$

(See [2] for more details.) If, moreover, M has finite derivation type, in the proofs [4,8,9] that finite derivation type implies the property left FP_3 for monoids and groups, it is shown that there is a free left $\mathbf{Z}M$-module

$$F_3 = \bigoplus_{\sigma \in \underline{\beta}^3} \mathbf{Z}M\sigma$$

with $\underline{\beta}^3 = B$ and an exact sequence

$$F_3 \to F_2 \to F_1 \to F_0 \to \mathbf{Z} \to 0.$$

3. Definition of homological finite derivation type

In this section we define a homological version of finite derivation type for all dimensions. To do this, we start from homological information and construct a graph resembling the graph X. We will work with bimodules throughout, to illustrate both the left and right actions together;

however, all of the discussion in the remainder of the paper can be done for left or right modules only, also.

Suppose that M is a monoid and that $\partial_n : F_n \to F_{n-1}$ is a homomorphism of $(\mathbf{Z}M, \mathbf{Z}M)$-bimodules. Suppose moreover that F_n is a free $(\mathbf{Z}M, \mathbf{Z}M)$-bimodule, and choose a basis $\underline{\beta}^n$. We can write

$$F_n = \bigoplus_{\sigma \in \underline{\beta}^n} \mathbf{Z}M\sigma M.$$

Let $\beta^n := \{m\sigma m' | m, m' \in M, \sigma \in \underline{\beta}^n\}$ be the corresponding $(\mathbf{Z}, \mathbf{Z})$-bimodule basis. As in the definition of finite derivation type, we associate a graph with this data, and study relations among the paths in this graph. Eventually the homomorphisms ∂_n we will consider will be the boundary homomorphisms of a resolution

$$F_n \overset{\partial_n}{\to} F_{n-1} \to \cdots \to F_0 \overset{\partial_0}{\to} \mathbf{Z} \to 0.$$

In this case, for ease of notation, we write $F_{-1} = \mathbf{Z}$.

Definition (Associated graph Γ_n). This is the graph whose vertices and edges are given by:
(1) Vertices: $V(\Gamma_n) := F_{n-1}$.
(2) Edges: $E(\Gamma_n) := \{(x, \sigma, y, \varepsilon) | x, y \in F_{n-1}, \sigma \in \beta^n, \varepsilon = \pm 1, \partial_n \sigma = (y - x)\}$.
(3) $\iota, \tau : E(\Gamma_n) \to V(\Gamma_n)$ are defined by (e denotes $(x, \sigma, y, \varepsilon)$):

$$\iota(e) := \begin{cases} x, & \text{for } \varepsilon = 1 \\ y, & \text{for } \varepsilon = -1 \end{cases} \qquad \tau(e) := \begin{cases} y, & \text{for } \varepsilon = 1 \\ x, & \text{for } \varepsilon = -1 \end{cases}$$

Note that $\partial_n \sigma = \varepsilon(\tau(e) - \iota(e))$.
(4) $()^{-1} : E(\Gamma_n) \to E(\Gamma_n)$ is given by $(x, \sigma, y, \varepsilon)^{-1} := (x, \sigma, y, -\varepsilon)$.

As noted above, the definition of Γ_n can also be applied to a homomorphism of left $\mathbf{Z}M$-modules, with $\underline{\beta}^n$ the basis of F_n as a free left $\mathbf{Z}M$-module, and $\beta^n := \{m\sigma | m \in M, \sigma \in \underline{\beta}^n\}$ the corresponding left $\mathbf{Z}$-module basis, in that case. Similarly, Γ_n can be defined for a homomorphism of right modules.

Note that if the monoid M has a finite presentation $\mathcal{P} = \langle A \mid R \rangle$, Sec. 2.2 describes an associated exact sequence of left $\mathbf{Z}M$-modules. The boundary map $\partial_2 : F_2 \to F_1$ in Sec. 2.2 corresponds to the same dimensional information as the graph X associated to the presentation $\mathcal{P}$, but gives rise to a graph Γ_2 which differs from X. In particular, the vertices of Γ_2 are elements of

$$F_1 = \bigoplus_{\sigma \in A} \mathbf{Z}M\sigma,$$

and the vertices of X are the elements of A^*.

Let $P(\Gamma_n)$ be the set of paths, or *homological derivations*, in Γ_n. If $x \in V(\Gamma_n) = F_{n-1}$, let (x) denote the constant path at the vertex x. For $p = e_1 \circ \cdots \circ e_k \in P(\Gamma_n)$, $\iota(p) := \iota(e_1)$, $\tau(p) := \tau(e_k)$, and $p^{-1} := e_1^{-1} \circ \cdots \circ e_k^{-1}$.

Just as for the graph X in Section 2.2, when $M = G$ is a finitely presented group the paths in Γ_n also have the following topological meaning. If Y is a $K(G, 1)$, then $C_*(\widetilde{Y})$, the augmented cellular chain complex for $\widetilde{Y}$, gives a resolution of $\mathbf{Z}$ by free left $\mathbf{Z}G$-modules. Choose a lift of each n-cell of Y in $\widetilde{Y}$; this gives a free left $\mathbf{Z}G$-module basis for $C_n(\widetilde{Y})$. The paths in the graph Γ_n

constructed from this data correspond essentially to formal sums of n-disks in $\widetilde{Y}$. Similarly, if Y is a $K(G \times G, 1)$, then $C_*(\widetilde{Y})$ gives rise to a resolution of $\mathbf{Z}$ by free $(\mathbf{Z}G, \mathbf{Z}G)$-bimodules, and paths in Γ_n again correspond essentially to formal sums of n-disks in $\widetilde{Y}$.

Definition (Action of M on $P(\Gamma_n)$). Given $m, m' \in M$ and $e = (x, \sigma, y, \varepsilon) \in E(\Gamma_n)$, we set

$$me := (mx, m\sigma, my, \varepsilon) \quad \text{and} \quad em := (xm, \sigma m, ym, \varepsilon),$$

which are edges in $E(\Gamma_n)$. Given a path $p = e_1 \circ \cdots \circ e_k \in P(\Gamma_n)$, we set

$$mp := (me_1) \circ \cdots \circ (me_k) \quad \text{and} \quad pm := (e_1 m) \circ \cdots \circ (e_k m),$$

which are paths in $P(\Gamma_n)$.

Definition (Addition in $P(\Gamma_n)$). Given $x, y \in F_{n-1}$ and $e = (x_1, \sigma, y_1, \varepsilon) \in E(\Gamma_n)$, we set

$$x + e := (x + x_1, \sigma, x + y_1, \varepsilon) \quad \text{and} \quad e + x := (x_1 + x, \sigma, y_1 + x, \varepsilon),$$

which are edges in $E(\Gamma_n)$. Given a path $p = e_1 \circ \cdots \circ e_k \in P(\Gamma_n)$, we set

$$x + p := (x + e_1) \circ \cdots \circ (x + e_k) \quad \text{and} \quad p + x := (e_1 + x) \circ \cdots \circ (e_k + x),$$

which are again paths in $P(\Gamma_n)$. Finally, given $p, q \in P(\Gamma_n)$, we set

$$p + q := (p + \iota(q)) \circ (\tau(p) + q).$$

Note that in the above definition, $x + p = p + x$. Also, if $p = (x)$ and $q = (y)$ are constant paths with $x, y \in F_{n-1}$, then $p + q = (x + y)$.

Definition (Negation in $P(\Gamma_n)$). Define negation in $P(\Gamma_n)$ by

$$-p = p^{-1}$$

for any $p \in P(\Gamma_n)$.

Definition ($P^{(b)}(\Gamma_n)$).

$$P^{(b)}(\Gamma_n) := \{(p, q) | p, q \in P(\Gamma_n),\ \iota(p) - \iota(q) = \tau(p) - \tau(q)\}$$

Definition (D_n, I_n, J_n).

$$\begin{aligned} D_n &:= \{(p + q, q + p) | p, q \in P(\Gamma_n)\} \\ I_n &:= \{(p \circ p^{-1}, (0)) | p \in P(\Gamma_n)\} \\ J_n &:= \{(p, p + x) | p \in P(\Gamma_n), x \in F_{n-1}\}. \end{aligned}$$

Definition (b-homology relation). A b-homology relation on $P(\Gamma_n)$ is an equivalence relation $\approx\ \subseteq P^{(b)}(\Gamma_n)$ such that:
(1) $D_n \cup I_n \cup J_n \subseteq\ \approx$.
(2) If $m, m' \in M$ and $p \approx q$, then $mp \approx mq$ and $pm' \approx qm'$.
(3) If $r, s \in P(\Gamma_n)$ and $p \approx q$, then $r + p \approx r + q$ and $p + s \approx q + s$.

For any set $B \subseteq P^{(b)}(\Gamma_n)$, the smallest possible b-homology relation containing B will be called the homology relation generated by B and denoted $\approx_B$.

Definition (n-dimensional homological finite derivation type). The monoid M has n-dimensional homological finite derivation type, or type $HFDT_n$, if there is an exact sequence

$$F_{n-1} \to F_{n-2} \to \cdots \to F_3 \to F_2 \to F_1 \to F_0 \to \mathbf{Z} \to 0$$

of free $(\mathbf{Z}M, \mathbf{Z}M)$-bimodules such that for every $i \geq 0$, there is a finite set $B_i \subseteq P^{(b)}(\Gamma_i)$ for which the b-homology relation generated by B_i is all of $P^{(b)}(\Gamma_i)$.

Note that, when applied to a monoid M, this homological definition does not require M to be finitely presented. We can similarly define notions of *left* $HFDT_n$ and *right* $HFDT_n$ by replacing the bimodules above by left or right $\mathbf{Z}M$-modules and redefining the b-homology relation to include only one-sided M-actions.

As noted in Sec. 2.1, the homological finiteness condition (left, right, or bi-) FP_n is equivalent to the condition that, for every partial finitely generated free (or projective) resolution $F_k \to \cdots \to F_0 \to \mathbf{Z} \to 0$ of (left, right, or bi-, resp.) $\mathbf{Z}M$-modules with $k < n$, $\ker\{F_k \to F_{k-1}\}$ is finitely generated. We can define a similar condition in the framework of homological finite derivation type.

Definition (Z_n). The monoid M has type Z_n if for every partial resolution

$$F_k \overset{\partial_k}{\to} F_{k-1} \to \cdots \to F_1 \to F_0 \to \mathbf{Z} \to 0$$

of finite rank free $(\mathbf{Z}M, \mathbf{Z}M)$-bimodules with $k < n$, there is a finite set $B \subseteq P^{(b)}(\Gamma_k)$ for which the b-homology relation generated by B is all of $P^{(b)}(\Gamma_k)$.

As mentioned above, we can similarly define the corresponding properties of *left* and *right* Z_n using left or right $\mathbf{Z}M$-modules and using the b-homology relation restricted to the corresponding side.

4. $\ker(\partial_n)$ and pairs of paths

In this section, we form a bimodule from the set $P^{(b)}(\Gamma_n)$ of pairs of paths in Γ_n, and show (in Theorem 4.8) that this bimodule is isomorphic to $\ker(\partial_n)$.

Definition ($P^{(b)}(\Gamma_n)/\sim$). Define an equivalence relation on $P^{(b)}(\Gamma_n)$ by

$$(p, q) \sim (r, s) \iff p - r \approx_\emptyset q - s,$$

where $\emptyset$ denotes the empty set. Define an action of M, addition, and negation in the set of equivalence classes $P^{(b)}(\Gamma_n)/\sim$ to be the action, addition, and negation induced componentwise from those in $P(\Gamma_n)$. Extend the action linearly to an action of $\mathbf{Z}M$ on both sides. Define the element $\bar{0}$ in $P^{(b)}(\Gamma_n)/\sim$ to be the equivalence class $\bar{0} = [(0), (0)]$, where (0) is the constant path at the element $0 \in F_{n-1}$.

Proposition 4.1. *$P^{(b)}(\Gamma_n)/\sim$ is a $(\mathbf{Z}M, \mathbf{Z}M)$-bimodule.*

We will prove this proposition using a series of lemmas.

Lemma 4.2. *Addition and action in $P(\Gamma_n)$ are associative and distributive.*

Proof. Suppose that $p, q, r \in P(\Gamma_n)$, $m, m' \in M$, and $x, y \in F_{n-1}$. Write $p = e_1 \circ \cdots \circ e_k$ and $e_i = (x_i, \sigma_i, y_i, \varepsilon_i)$.

Using associativity of the monoid action on F_n and F_{n-1} gives

$$\begin{aligned} m(m'e_i) &= m(m'x_i, m'\sigma_i, m'y_i, \varepsilon_i) = (m(m'x_i), m(m'\sigma_i), m(m'y_i), \varepsilon_i) \\ &= ((mm')x_i, (mm')\sigma_i, (mm')y_i, \varepsilon_i) = (mm')e_i. \end{aligned}$$

Therefore

$$\begin{aligned} m(m'p) &= m((m'e_1) \circ \cdots \circ (m'e_k)) = (m(m'e_1)) \circ \cdots \circ (m(m'e_k)) \\ &= ((mm')e_1) \circ \cdots \circ ((mm')e_k) = (mm')p. \end{aligned}$$

Similarly, $(mp)m' = m(pm')$ and $p(mm') = (pm)m'$, so the monoid action is associative and distributive.

Using associativity of addition in F_{n-1} gives

$$\begin{aligned} (x+y) + e_i &= (x+y) + (x_i, \sigma_i, y_i, \varepsilon_i) = ((x+y) + x_i, \sigma_i, (x+y) + y_i, \varepsilon_i) \\ &= (x + (y + x_i), \sigma_i, x + (y + y_i), \varepsilon_i) = x + (y + e_i) \end{aligned}$$

so

$$\begin{aligned} (x+y) + p &= (x+y) + e_1 \circ \cdots \circ e_k = (x+y) + e_1 \circ \cdots \circ (x+y) + e_k \\ &= x + (y + e_1) \circ \cdots \circ x + (y + e_k) = x + (y + p). \end{aligned}$$

Then $(x+p)+q = (x+p+\iota(q)) \circ (\tau(x+p)+q) = (x+p+\iota(q)) \circ (x+\tau(p)+q) = x + [(p+\iota(q)) \circ (\tau(p)+q)] = x + (p+q)$. Finally,

$$\begin{aligned} (p+q)+r &= [(p+\iota(q)) \circ (\tau(p)+q)] + r \\ &= ([(p+\iota(q)) \circ (\tau(p)+q)] + \iota(r)) \circ (\tau[(p+\iota(q)) \circ (\tau(p)+q)] + r) \\ &= [(p+\iota(q)+\iota(r)) \circ (\tau(p)+q+\iota(r))] \circ (\tau(p)+\tau(q)+r) \\ &= (p+\iota[(q+\iota(r)) \circ (\tau(q)+r)]) \circ (\tau(p) + [(q+\iota(r)) \circ (\tau(q)+r)]) \\ &= p + [(q+\iota(r)) \circ (\tau(q)+r)] = p + (q+r), \end{aligned}$$

giving associativity for addition.

Using the distributive property for F_n and F_{n-1} gives

$$\begin{aligned} m(e_i + x) &= m(x_i + x, \sigma_i, y_i + x, \varepsilon_i) = (m(x_i + x), m\sigma_i, m(y_i + x), \varepsilon_i) \\ &= ((mx_i + mx), m\sigma_i, (my_i + mx), \varepsilon_i) = me_i + mx \end{aligned}$$

and similarly $m(y + e_i) = my + me_i$. Also, it is straightforward to check $\iota(mq) = m\iota(q)$ and $\tau(mp) = m\tau(p)$. Then

$$\begin{aligned} m(p+q) &= m[(p+\iota(q)) \circ (\tau(p)+q)] \\ &= [m(p+\iota(q))] \circ [m(\tau(p)+q)] = (mp+\iota(mq)) \circ (\tau(mp)+mq) = mp+mq. \end{aligned}$$

The remaining proof of distributivity on the other side is similar. $\square$

Lemma 4.3. *Suppose that $p, q \in P(\Gamma_n)$ and $x \in F_{n-1}$. Then*

(i) $p + q \approx_\emptyset q + p$.

(ii) $p - p \approx_\emptyset (0)$.

(iii) $p + x \approx_\emptyset p$.

(iv) $-(p + q) \approx_\emptyset -q - p$.

(v) *If* $p \approx_\emptyset q$, *then* $-p \approx_\emptyset -q$.

(vi) *If* $\tau(p) = \iota(q)$, *then* $p \circ q \approx_\emptyset p + q$. *In particular, if* $p = e_1 \circ \cdots \circ e_n \in P(\Gamma_n)$ *then* $p \approx_\emptyset e_1 + \cdots + e_n$.

Proof. The results in (i), (ii), and (iii) follow directly from the fact that $D_n \cup I_n \cup J_n \subseteq \approx_\emptyset$. If $p, q \in P(\Gamma_n)$ and $x \in F_{n-1}$, then $(p + x)^{-1} = p^{-1} + x$, so

$$\begin{aligned}-(p+q) &= -[(p+\iota(q)) \circ (\tau(p)+q)] = (\tau(p)+q)^{-1} \circ (p+\iota(q))^{-1} \\ &= (\tau(p)+q^{-1}) \circ (p^{-1}+\iota(q)) = (\iota(p^{-1})+q^{-1}) \circ (p^{-1}+\tau(q^{-1})) \\ &= (q^{-1}+\iota(p^{-1})) \circ (\tau(q^{-1})+p^{-1}) = -p-q.\end{aligned}$$

If $p \approx_\emptyset q$, then $I_n \cup J_n \subseteq \approx_\emptyset$ and Part (3) of the definition of a b-homology relation give that $-p \approx_\emptyset -p + (0) \approx_\emptyset -p + q - q \approx_\emptyset -p + p - q \approx_\emptyset (0) - q \approx_\emptyset -q$.

If $\tau(p) = \iota(q)$, then the fact that $J_n \subseteq \approx_\emptyset$ implies

$$\begin{aligned}p+q &= (p+\iota(q)) \circ (\tau(p)+q) = (p+\iota(q)) \circ (\iota(q)+q) \\ &= (p+\iota(q)) \circ (q+\iota(q)) = (p \circ q) + \iota(q) \\ &\approx_\emptyset p \circ q.\end{aligned}$$

□

Proof of Proposition 4.1. First we show that addition, negation, and scalar multiplication are well-defined. Suppose that $[p, q]$ and $[r, s]$ are elements of $P^{(b)}(\Gamma_n)/\sim$, where $(p, q), (r, s) \in P^{(b)}(\Gamma_n)$. Then $\iota(p) - \iota(q) = \tau(p) - \tau(q)$ and $\iota(r) - \iota(s) = \tau(r) - \tau(s)$, so

$$\begin{aligned}\iota(p+r) - \iota(q+s) &= \iota[(p+\iota(r)) \circ (\tau(p)+r)] - \iota[(q+\iota(s)) \circ (\tau(q)+s)] \\ &= \iota(p) + \iota(r) - (\iota(q)+\iota(s)) = \tau(p) + \tau(r) - (\tau(q)+\tau(s)) \\ &= \tau(p+r) - \tau(q+s).\end{aligned}$$

Therefore $(p + r, q + s) \in P^{(b)}(\Gamma_n)$ and $[p, q] + [r, s] := [p + r, q + s] \in P^{(b)}(\Gamma_n)/\sim$. Suppose next that $[p, q] = [p', q']$ and $[r, s] = [r', s']$ are elements of $P^{(b)}(\Gamma_n)/\sim$. Then $p - p' \approx_\emptyset q - q'$ and $r - r' \approx_\emptyset s - s'$, so Part (3) of the definition of a b-homology relation says that $(p - p') + (r - r') \approx_\emptyset (q - q') + (r - r') \approx_\emptyset (q - q') + (s - s')$. Then $[p, q] + [r, s] = [p', q'] + [r', s']$ and addition is well-defined.

Also, $\iota(p^{-1}) - \iota(q^{-1}) = \tau(p) - \tau(q) = \iota(p) - \iota(q) = \tau(p^{-1}) - \tau(q^{-1})$, so $-[p, q] := [-p, -q] \in P^{(b)}(\Gamma_n)/\sim$. If $[p, q] = [p', q']$, then $p - p' \approx_\emptyset q - q'$. Lemma 4.3 (iv) and (v) say $p' - p \approx_\emptyset q' - q$, so $-[p, q] = -[p', q']$ and negation is well-defined.

If $m, m' \in M$, then

$$\begin{aligned}\iota(mpm') - \iota(mqm') &= m\iota(p)m' - m\iota(q)m' = m(\iota(p) - \iota(q))m' \\ &= m(\tau(p) - \tau(q))m' = m\tau(p)m' - m\tau(q)m' = \tau(mpm') - \tau(mqm')\end{aligned}$$

so $m[p,q]m' := [mpm', mqm'] \in P^{(b)}(\Gamma_n)/\sim$. The fact that scalar multiplication is well-defined then follows directly from Part (2) of the definition of a b-homology relation.

Associativity of addition in $P^{(b)}(\Gamma_n)/\sim$ follows directly from associativity of addition in $P(\Gamma_n)$ (Lemma 4.2), and commutativity follows from Lemma 4.3(i) and (v). Lemma 4.3(ii) and (iii) imply that the additive identity in $P^{(b)}(\Gamma_n)/\sim$ is $\bar{0}$, and $-[p,q]$ is the additive inverse of $[p,q]$. Finally, associativity of the $\mathbf{Z}M$ actions and the distributive laws follow from the definition of the $\mathbf{Z}M$ actions and Lemma 4.2. □

In order to prove that the bimodule $P^{(b)}(\Gamma_n)/\sim$ is isomorphic to $\ker(\partial_n)$, we will need some further notation to construct the homomorphism.

Definition ($c : P(\Gamma_n) \to F_n$). For a vertex $x \in F_{n-1}$, set $c((x)) = 0$. If $e = (x, \sigma, y, \varepsilon) \in E(\Gamma_n)$, set $c(e) = \varepsilon\sigma$. Finally, for any path $p = e_1 \circ \cdots \circ e_k \in P(\Gamma_n)$, set

$$c(p) = \sum_{i=1}^{k} c(e_i).$$

Lemma 4.4. *Suppose $p, q \in P(\Gamma_n)$, $m, m' \in M$, and $\varepsilon = \pm 1$.*
(i) *$\partial_n(c(p)) = \tau(p) - \iota(p)$.*
(ii) *$c(\varepsilon m p m') = \varepsilon m c(p) m'$ and $c(p+q) = c(p) + c(q)$.*
(iii) *If $p \approx_\emptyset q$, then $c(p) = c(q)$.*

Proof. If $p \in P(\Gamma_n)$, write $p = e_1 \circ \cdots \circ e_k$ with $e_i = (x_1, \sigma_i, y_i, \varepsilon_i)$. Then

$$\partial_n(c(p)) = \partial_n(\sum_{i=1}^{k} \varepsilon_i\sigma_i) = \sum_{i=1}^{k} \varepsilon_i \partial_n(\sigma_i) = \sum_{i=1}^{k} \tau(e_i) - \iota(e_i) = \tau(p) - \iota(p),$$

giving (i). Part (ii) follows directly from the definition of the map c.

If $p \approx_\emptyset q$, then there is a sequence $p = z_1 \approx_\emptyset z_2 \approx_\emptyset \cdots \approx_\emptyset z_l = q$ with, at each step, $z_i = r_i + s_i + t_i$, $z_{i+1} = r_i + u_i + t_i$, $s_i = \varepsilon_i m_i v_i m_i'$, and $u_i = \varepsilon_i m_i w_i m_i'$, where $r_i, t_i, v_i, w_i \in P(\Gamma_n)$, $m_i, m_i' \in M$, $\varepsilon_i = \pm 1$, and either (v_i, w_i) or (w_i, v_i) is in $D_n \cup I_n \cup J_n$. It follows directly from the definitions of c, D_n, I_n, and J_n that $c(v_i) = c(w_i)$ for each i. Then

$$c(z_i) = c(r_i) + \varepsilon_i m_i c(v_i) m_i' + c(t_i) = c(r_i) + \varepsilon_i m_i c(w_i) m_i' + c(t_i) = c(z_{i+1})$$

for each i, so $c(p) = c(q)$. □

Definition ($\varphi : P^{(b)}(\Gamma_n) \to F_n$). For any pair $(p,q) \in P^{(b)}(\Gamma_n)$, define $\varphi((p,q)) = c(p) - c(q)$.

Proposition 4.5. $\mathrm{im}(\varphi) \subseteq \ker(\partial_n)$ *and φ induces a $(\mathbf{Z}M, \mathbf{Z}M)$-bimodule homomorphism*

$$\overline{\varphi} : P^{(b)}(\Gamma_n)/\sim \;\to \ker(\partial_n)$$

giving the commutative diagram

$$\begin{array}{ccc} P^{(b)}(\Gamma_n) & \xrightarrow{\varphi} & \ker(\partial_n) \\ \downarrow & & \| \\ P^{(b)}(\Gamma_n)/\sim & \xrightarrow{\overline{\varphi}} & \ker(\partial_n). \end{array}$$

Proof. For $(p,q) \in P^{(b)}(\Gamma_n)$, using Lemma 4.4(i),

$$\partial_n(\varphi((p,q))) = \partial_n(c(p)) - \partial_n(c(q)) = \tau(p) - \iota(p) - (\tau(q) - \iota(q)) = 0.$$

Therefore $P^{(b)}(\Gamma_n)$ is exactly the set of pairs of paths in Γ_n for which $\partial_n \circ \varphi$ acts by 0, and $\mathrm{im}(\varphi) \subseteq \ker(\partial_n)$.

Suppose that $[p,q],[r,s] \in P^{(b)}(\Gamma_n)/\sim$ and $[p,q]=[r,s]$. Then $p-r \approx_\emptyset q-s$, so Lemma 4.4(iii) says that $c(p-r) = c(q-s)$. Lemma 4.4(ii) says that $c(p-r) = c(p)-c(r)$, so $c(p)-c(r) = c(q)-c(s)$ and $\varphi((p,q)) = c(p)-c(q) = c(r)-c(s) = \varphi((r,s))$. Then for the map $\overline{\varphi}([p,q]) := \varphi((p,q))$ we have $\overline{\varphi}([p,q]) = \overline{\varphi}([r,s])$ and $\overline{\varphi}$ is well-defined.

For any $[p,q],[r,s] \in P^{(b)}(\Gamma_n)/\sim$,

$$\begin{aligned}\overline{\varphi}([p,q]+[r,s]) &= \overline{\varphi}([p+r,q+s]) = c(p+r) - c(q+s)\\ &= c(p) - c(q) + c(r) - c(s) = \overline{\varphi}([p,q]) + \overline{\varphi}([r,s]).\end{aligned}$$

If $m, m' \in M$ and $\varepsilon = \pm 1$, then

$$\begin{aligned}\overline{\varphi}(\varepsilon m[p,q]m') &= \overline{\varphi}([\varepsilon mpm', \varepsilon mqm']) = c(\varepsilon mpm') - c(\varepsilon mqm')\\ &= \varepsilon mc(p)m' - \varepsilon mc(q)m' = \varepsilon m\overline{\varphi}([p,q])m'.\end{aligned}$$

Therefore $\overline{\varphi}$ is also a bimodule homomorphism. ☐

Proposition 4.6. *$\overline{\varphi}$ is injective.*

Proof. In view of Proposition 4.5, it suffices to show that $\ker(\overline{\varphi}) = 0$. Suppose $[(p,q)] \in P^{(b)}(\Gamma_n)/\sim$ with

$$\overline{\varphi}([(p,q)]) = \varphi((p,q)) = c(p) - c(q) = c(p-q) = 0.$$

Let $r := p - q$, and write $r = e_1 \circ \cdots \circ e_k$ where $e_i = (x_i, \sigma_i, y_i, \varepsilon_i)$; then $c(r) = \sum \varepsilon_i \sigma_i = 0$.

Suppose that r has at least one edge. Since F_n is **Z**-free on β^n, it follows from $\sum \varepsilon_i \sigma_i = 0$ that $k = 2k'$ for some $k' > 0$ and that there exists a permutation π of $\{1, \ldots, k\}$ such that $\varepsilon_i + \varepsilon_{\pi(i)} = 0$, $\sigma_i = \sigma_{\pi(i)}$, and $\pi(\pi(i)) = i$ for all i. By definition of edges in Γ_n, we have that $y_i - x_i = \partial_n \sigma_i = \partial_n \sigma_{\pi(i)} = y_{\pi(i)} - x_{\pi(i)}$. Let $t := x_{\pi(i)} - x_i = y_{\pi(i)} - y_i \in F_{n-1}$. Then

$$e_{\pi(i)} = (x_{\pi(i)}, \sigma_{\pi(i)}, y_{\pi(i)}, \varepsilon_{\pi(i)}) = (x_i + (x_{\pi(i)} - x_i), \sigma_i, y_i + (y_{\pi(i)} - y_i), -\varepsilon_i) = e_i^{-1} + t.$$

Lemmas 4.2 and 4.3 imply that

$$e_i + e_{\pi(i)} = e_i + (e_i^{-1} + t) \approx_\emptyset (e_i + e_i^{-1}) + t \approx_\emptyset e_i + e_i^{-1} \approx_\emptyset e_i \circ e_i^{-1} \approx_\emptyset (0).$$

Applying Lemmas 4.2 and 4.3 again along with Part (3) of the definition of b-homology relation gives

$$r = e_1 \circ \cdots \circ e_{2k'} \approx_\emptyset \sum_{j=1}^{2k'} e_j = \sum_i (e_i + e_{\pi(i)}) \approx_\emptyset (0)$$

where the last sum ranges over indices i with one index from every (two element) orbit of the permutation π.

Suppose now that $k = 0$ and r does not contain a single edge. In this case, also, we get $r \approx_\emptyset (0)$. Thus in both cases, $(0) \approx_\emptyset r = p - q$, and $p = p + (0) \approx_\emptyset p - q + q \approx_\emptyset q$. Then $p - (0) \approx_\emptyset q - (0)$, so $(p,q) \sim ((0),(0))$ and $[(p,q)] = \bar{0}$. This completes the proof of injectivity. ☐

Proposition 4.7. *$\overline{\varphi} : P^{(b)}(\Gamma_n)/\sim \;\to \ker(\partial_n)$ is surjective.*

Proof. Suppose that $z \in F_n$ and $z \neq 0$. Then z can be written (not necessarily uniquely) as

$$z = \sum_{i=1}^{\ell} \lambda_i \sigma_i, \qquad \lambda_i = \pm 1, \qquad \sigma_i \in \beta^n$$

with $\ell \geq 1$. Suppose moreover that $z \in \ker(\partial_n)$.

For $1 \leq i \leq \ell$, define edges

$$e_i := \begin{cases} (N_{i-1}, \sigma_i, N_{i-1} + \partial_n(\sigma_i), 1), & \text{for } \lambda_i = 1 \\ (N_{i-1} - \partial_n(\sigma_i), \sigma_i, N_{i-1}, -1), & \text{for } \lambda_i = -1 \end{cases}$$

where $N_0 := 0$ and

$$N_i := \sum_{j=1}^{i} \lambda_j \partial_n(\sigma_j).$$

Then $\iota(e_i) = N_{i-1}$, $\tau(e_i) = N_i$, and $c(e_i) = \lambda_i \sigma_i$. Therefore these edges form a path $p := e_1 \circ \cdots \circ e_\ell$ with $\iota(p) = 0$, $\tau(p) = N_\ell = \partial_n(z) = 0$, and

$$c(p) = \sum_{i=1}^{\ell} c(e_i) = \sum_{i=1}^{\ell} \lambda_i \sigma_i = z.$$

Define another path q to be the constant path $q := (0)$ at $0 \in F_{n-1}$, so that $\iota(q) = \tau(q) = 0$ and $c(q) = 0$. Thus $(p, q) \in P^{(b)}(\Gamma_n)$, and

$$\varphi(p, q) = c(p) - c(q) = z - 0 = z.$$

Therefore $\overline{\varphi}([p, q]) = z$, as desired. □

The following theorem now follows directly from Propositions 4.1, 4.5, 4.6, and 4.7.

Theorem 4.8. $\ker(\partial_n)$ *and* $P^{(b)}(\Gamma_n)/\sim$ *are isomorphic* $(\mathbf{Z}M, \mathbf{Z}M)$*-bimodules.*

A straightforward modification of the definition of $P^{(b)}(\Gamma_n)/\sim$ and the proofs in this section also leads to the following.

Corollary 4.9. *If* ∂_n *is a homomorphism of left (respectively right)* $\mathbf{Z}M$*-modules, then* $\ker(\partial_n)$ *and* $P^{(b)}(\Gamma_n)/\sim$ *are isomorphic left (respectively right)* $\mathbf{Z}M$*-modules.*

5. The main theorem: $HFDT_n$ and FP_n

In this section we prove the following.

Theorem 5.1. *A group* G *has type* $HFDT_n$ *if and only if* G *has homological type* FP_n*. A monoid* M *has type* $HFDT_n$ *if and only if* M *has type bi-* FP_n*.*

Since (bi-) FP_n is a property of a monoid, rather than simply a property of a resolution, it follows that the property $HFDT_n$ is also a monoid property. We will prove this theorem using several propositions.

Lemma 5.2. *Suppose that* $p, q \in P(\Gamma_n)$. *Then*
(i) $[p,p] = \bar{0}$.
(ii) *If* (p,q) *or* (q,p) *is in* $D_n \cup I_n \cup J_n$, *then* $[p,q] = \bar{0}$.

Proof. For $p \in P(\Gamma_n)$, $p - (0) \approx_\emptyset p - (0)$, so $[p,p] = [(0),(0)] = \bar{0}$.

For $(p+q, q+p) \in D_n$, it follows from Lemma 4.3(i) that $p + q - (0) \approx_\emptyset q + p - (0)$, so $[p+q, q+p] = \bar{0}$. The other parts of (ii) follow from Lemma 4.3(ii)-(iii). ◻

Proposition 5.3. $P^{(b)}(\Gamma_n)$ *is finitely generated by a b-homology relation if and only if* $P^{(b)}(\Gamma_n)/\sim$ *is a finitely generated* $(\mathbf{Z}M, \mathbf{Z}M)$*-bimodule.*

Proof. Suppose first that $P^{(b)}(\Gamma_n)$ is finitely generated by a b-homology relation, so $P^{(b)}(\Gamma_n) = \approx_B$ for some finite subset $B \subseteq P^{(b)}(\Gamma_n)$. Let $[p,q] \in P^{(b)}(\Gamma_n)/\sim$. Since $(p,q) \in P^{(b)}(\Gamma_n)$, $p \approx_B q$, so there is a sequence $p = z_1 \approx_B z_2 \approx_B \cdots \approx_B z_l = q$ with, at each step, $z_i = r_i + s_i + t_i$, $z_{i+1} = r_i + u_i + t_i$, $s_i = \varepsilon_i m_i v_i m_i'$, and $u_i = \varepsilon_i m_i w_i m_i'$, where $r_i, t_i, v_i, w_i \in P(\Gamma_n)$, $m_i, m_i' \in M$, $\varepsilon_i = \pm 1$, and either (v_i, w_i) or (w_i, v_i) is in $B \cup D_n \cup I_n \cup J_n$. Then in $P^{(b)}(\Gamma_n)/\sim$ (using Lemma 5.2)

$$\begin{aligned}
[p,q] &= [p,p] + [(0), q-p] = \bar{0} + \sum_{i=1}^{l} [(0), z_{i+1} - z_i] \\
&= \sum_{i=1}^{l} \bar{0} + [(0), z_{i+1} - z_i] = \sum_{i=1}^{l} [z_i, z_i] + [(0), z_{i+1} - z_i] \\
&= \sum_{i=1}^{l} [z_i, z_{i+1}] = \sum_{i=1}^{l} [r_i, r_i] + \varepsilon_i m_i [v_i, w_i] m_i' + [t_i, t_i] \\
&= \sum_{i=1}^{l} \bar{0} + \varepsilon_i m_i [v_i, w_i] m_i' + \bar{0} = \bar{0} + \sum{}' \varepsilon_i m_i [v_i, w_i] m_i'
\end{aligned}$$

where the last sum ranges over only the indices i for which either (v_i, w_i) or (w_i, v_i) is in B. Then the set

$$C := \{[v,w] \mid \text{either } (v,w) \text{ or } (w,v) \text{ is in } B\}$$

is a finite set which generates $P^{(b)}(\Gamma_n)/\sim$ as a $(\mathbf{Z}M, \mathbf{Z}M)$-bimodule.

Next suppose that $P^{(b)}(\Gamma_n)/\sim$ is finitely generated by a subset C as a $(\mathbf{Z}M, \mathbf{Z}M)$-bimodule. Let B be a finite subset of $P^{(b)}(\Gamma_n)$ consisting of one representative of each element of C. Let (p,q) be an arbitrary element of $P^{(b)}(\Gamma_n)$. Then

$$[p,q] = \sum_{i=1}^{l} \varepsilon_i m_i [p_i, q_i] m_i' = \sum_{i=1}^{l} [\varepsilon_i m_i p_i m_i', \varepsilon_i m_i q_i m_i']$$

for some $m_i, m_i' \in M$, $\varepsilon_i = \pm 1$, and $(p_i, q_i) \in B$. So $p - \sum_{i=1}^{l} \varepsilon_i m_i p_i m_i' \approx_\emptyset q - \sum_{i=1}^{l} \varepsilon_i m_i q_i m_i'$. Since $p_i \approx_B q_i$ for each index i, $\sum_{i=1}^{l} \varepsilon_i m_i p_i m_i' \approx_B \sum_{i=1}^{l} \varepsilon_i m_i q_i m_i'$. It follows from Part (3) of the definition of a b-homology relation that $p \approx_B q$, so $(p,q) \in \approx_B$. Therefore the finite set B generates all of $P^{(b)}(\Gamma_n)$ as a b-homology relation. ◻

Proof of theorem 5.1. Suppose that M has type $HFDT_n$. Then there is a resolution

$$F_{n-1} \to F_{n-2} \to \cdots \to F_3 \to F_2 \to F_1 \to F_0 \to \mathbf{Z} \to 0$$

of free $(\mathbf{Z}M, \mathbf{Z}M)$-bimodules such that for every $0 \leq i \leq n-1$, there is a finite set $B_i \subseteq P^{(b)}(\Gamma_i)$ for which the b-homology relation generated by B_i is all of $P^{(b)}(\Gamma_i)$. Proposition 5.3 and Theorem 4.8 then say that $\ker(\partial_i)$ is finitely generated for each i. For $0 < i \leq n-1$, $\operatorname{im}(\partial_i) = \ker(\partial_{i-1})$, so both $\operatorname{im}(\partial_i)$ and $\ker(\partial_i)$ are finitely generated. If $i = 0$, then $\operatorname{im}(\partial_0) = \mathbf{Z}$, so again both $\operatorname{im}(\partial_0)$ and $\ker(\partial_0)$ are finitely generated. Construct a set S_i in F_i which is the union of a finite set of generators for $\ker(\partial_i)$ together with a set consisting of a preimage (under the map ∂_i) for each element in a finite set of generators for $\operatorname{im}(\partial_i)$. Then S_i is a finite set of generators for F_i. Hence F_i is a free $(\mathbf{Z}M, \mathbf{Z}M)$-bimodule of finite rank for each $0 \leq i \leq n-1$, and $\ker(\partial_{n-1})$ is finitely generated. Therefore M has type bi- FP_n.

Now suppose that M has type bi- FP_n. Then there is a partial finitely generated projective resolution of the integers by bimodules over the integral monoid ring of length n. With this resolution, for each $0 \leq i \leq n-1$, $\ker(\partial_i) = \operatorname{im}(\partial_{i+1})$. So $\ker(\partial_i)$ is the image of a finitely generated bimodule, and hence is also finitely generated, when $0 \leq i \leq n-1$. Then Proposition 5.3 and Theorem 4.8 say that $P^{(b)}(\Gamma_i)$ is finitely generated by a b-homology relation for each $0 \leq i \leq n-1$, and therefore M has type $HFDT_n$.

If G is a group, then the proof above together with the equivalence of the property bi- FP_n and FP_n in Proposition 2.1 show that G has type $HFDT_n$ if and only if G has type FP_n. This completes the proof of Theorem 5.1. □

The following corollary results from a straightforward modification of the proofs above.

Corollary 5.4. *A monoid or group M has type left (respectively right) $HFDT_n$ if and only if M is of left (respectively right) homological type FP_n.*

We can apply Theorem 5.1 to show that the other finiteness condition Z_n defined in Sec. 3 is also equivalent to $HFDT_n$.

Theorem 5.5. *The following conditions are equivalent for any monoid or group M and any integer $n \geq 0$.*
(i) *M has type bi- FP_n (if M is a group, M has type FP_n).*
(ii) *M has type $HFDT_n$.*
(iii) *M has type Z_n.*

Proof. It follows directly from Theorem 5.1 that (i) implies (ii) and (ii) implies (i). Next suppose (i) holds, and suppose also that $F_k \to \cdots F_0 \to \mathbf{Z} \to 0$ is a resolution consisting of finite rank free bimodules with $k < n$. Then since M has type bi- FP_n, $\ker(\partial_k)$ is finitely generated. Theorem 4.8 then says that $P^{(b)}(\Gamma_k)/\sim$ is finitely generated, and Proposition 5.3 applies to say that in this case $P^{(b)}(\Gamma_k)$ is finitely generated by a b-homology relation. Therefore (i) implies (iii).

Finally, suppose that (iii) holds, and suppose that $F_k \to \cdots F_0 \to \mathbf{Z} \to 0$ is a resolution consisting of finite rank free bimodules with $k < n$. Since M satisfies the property Z_n, $P^{(b)}(\Gamma_k)$ is finitely generated by a b-homology relation. Applying Proposition 5.3 and Theorem 4.8 in the opposite order shows that then $\ker(\partial_k)$ is finitely generated. Therefore (iii) implies (i) also. □

The following also results from a straightforward modification of the proofs above.

Corollary 5.6. *The following conditions are equivalent for any monoid or group M and any integer $n \geq 0$.*

i) *M has type left (resp. right) FP_n.*

ii) *M has type left (resp. right) $HFDT_n$.*

iii) *M has type left (resp. right) Z_n.*

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SICS Swedish Institute of Computer Science, Box 1263, SE-164 29 Kista, Sweden, Email: `alonso@sics.se`

Department of Mathematics and Statistics, University of Nebraska, Lincoln, NE 68588-0323, USA, Email: `smh@math.unl.edu`