ALGORITHMS AND TOPOLOGY OF CAYLEY GRAPHS FOR GROUPS

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ABSTRACT. Autostackability for finitely generated groups is defined via a topological property of the associated Cayley graph which can be encoded in a finite state automaton. Autostackable groups have solvable word problem and an effective inductive procedure for constructing van Kampen diagrams with respect to a canonical finite presentation. A comparison with automatic groups is given. Another characterization of autostackability is given in terms of prefix-rewriting systems. Every group which admits a finite complete rewriting system or an asynchronously automatic structure with respect to a prefix-closed set of normal forms is also autostackable. As a consequence, the fundamental group of every closed 3-manifold with any of the eight possible uniform geometries is autostackable.

1. INTRODUCTION

A primary motivation for the definition of the class of automatic groups is to make computing the word problem for 3-manifold groups tractable; however, in their introduction of the theory of automatic groups, Epstein, *et. al.* [10] showed that the fundamental group of a closed 3-manifold having Nil or Sol geometry is not automatic. Brady [1] showed that there are Sol geometry groups that do not belong to the wider class of asynchronously automatic groups. Bridson and Gilman [4] further relaxed the language theoretic restriction on the associated normal forms, replacing regular with indexed languages, and showed that every 3-manifold group has an asynchronous combing with respect to an indexed language. More recently, Kharlampovich, Khoussainov, and Miasnikov [22] have defined the class of Cayley automatic groups, extending the notion of an automatic structure (preserving the regular language restriction), but it is as yet unknown whether all Nil and Sol 3-manifold groups are Cayley automatic. In this paper we define the notion of autostackability for finitely generated groups using properties very closely related to automatic structures, that holds for 3-manifold groups of all uniform geometries.

Let G be a group with an inverse-closed finite generating set A, and let $\Gamma = \Gamma(G, A)$ be the associated Cayley graph. Let $\vec{E}$ be the set of directed

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edges; for each $g \in G$ and $a \in A$, let $e_{g,a}$ denote the directed edge of Γ with initial vertex g, terminal vertex ga, and label a. Let $\mathcal{N} \subset A^*$ be a set of normal forms for G over A; for each $g \in G$, we denote the normal form word representing g by y_g. Note that whenever we have an equality of words $y_g a = y_{ga}$ or $y_g = y_{ga} a^{-1}$, then there is a van Kampen diagram for the word $y_g a y_{ga}^{-1}$ that contains no 2-cells; in this case we call the edge $e_{g,a}$ *degenerate*. Let $\vec{E}_{\mathcal{N},d} = \vec{E}_d$ be the set of all degenerate directed edges, and let $\vec{E}_{\mathcal{N},r} = \vec{E}_r := \vec{E} \setminus \vec{E}_d$; we refer to elements of $\vec{E}_r$ as *recursive* edges.

Definition 1.1. *A group G with finite inverse-closed generating set A is* autostackable *if there are a set $\mathcal{N}$ of normal forms for G over A containing the empty word, a constant k, and a function $\phi : \mathcal{N} \times A \to A^*$ such that the following hold:*

(1) *The graph of the function ϕ,*

$$graph(\phi) := \{(y_g, a, \phi(y_g, a)) \mid g \in G, a \in A\},$$

is a synchronously regular language.

(2) *For each $g \in G$ and $a \in A$, the word $\phi(y_g, a)$ has length at most k and represents the element a of G, and:*

(2d) *If $e_{g,a} \in \vec{E}_{\mathcal{N},d}$, then the equality of words $\phi(y_g, a) = a$ holds.*

(2r) *The transitive closure $<_\phi$ of the relation $<$ on $\vec{E}_{\mathcal{N},r}$, defined by*

$e' < e_{g,a}$ whenever $e_{g,a}, e' \in \vec{E}_{\mathcal{N},r}$ and e' is on the directed path in Γ labeled $\phi(y_g, a)$ starting at the vertex g

is a strict well-founded partial ordering.

Removing the algorithmic property in (1), the group G is called *stackable* over the inverse-closed generating set A if property (2) holds for some normal form set $\mathcal{N}$ (containing the empty word), constant k, and function $\phi : \mathcal{N} \times A \to A^*$. In [6], the first two authors define and study the class of stackable groups. In [6, Lemma 1.5] they show that stackability implies that the finite set R_c of words of the form $\phi(y_g, a)a^{-1}$ (for $g \in G$ and $a \in A$) is a set of defining relators for G, and the set $\mathcal{N}$ of normal forms is closed under taking prefixes. Hence the set $\mathcal{N}$ uniquely determines a maximal tree in the Cayley graph Γ, consisting of the edges that lie on paths labeled by words in $\mathcal{N}$.

This leads to a topological description of the concept of autostackability. Let T be a maximal tree in Γ. For each $g \in G$ and $a \in A$, we view the two directed edges $e_{g,a}$ and $e_{ga,a^{-1}}$ of Γ to have a single underlying undirected edge in Γ. Let $\vec{P}$ be the set of all finite length directed edge paths in Γ. A *flow* function associated to T is a function $\Phi : \vec{E} \to \vec{P}$ satisfying the properties that:

(a) For each edge $e \in \vec{E}$, the path $\Phi(e)$ has the same initial and terminal vertices as e.

(b-d) If the undirected edge underlying e lies in the tree T, then $\Phi(e) = e$.

(b-r) The transitive closure $<_\Phi$ of the relation $<$ on $\vec{E}$, defined by
$e' < e$ whenever e' lies on the path $\Phi(e)$ and the undirected edges underlying both e and e' do not lie in T,
is a strict well-founded partial ordering.

That is, the map Φ fixes the edges lying in the tree T and describes a "flow" of the non-tree edges toward the tree (or toward the basepoint). A flow function is *bounded* if there is a constant k such that for all $e \in \vec{E}$, the path $\Phi(e)$ has length at most k.

For each element $g \in G$, let y_g be the unique word labeling a geodesic path in the tree T from the identity element 1 of G to g, and let $\mathcal{N}_T := \{y_g \mid g \in G\}$ be the corresponding set of normal forms. Let $\beta_T : \mathcal{N}_T \times A \to \vec{E}$ denote the natural bijection defined by $\beta_T(y_g, a) := e_{g,a}$, and let $\rho : \vec{P} \to A^*$ be the function that maps each directed path to the word labeling that path in Γ. The composition $\rho \circ \Phi \circ \beta_T : \mathcal{N} \times A \to A^*$ is part of a stackable structure for G over A, which we call the *induced stacking function.*Conversely, [6, Lemma 1.5] implies that given a stacking function $\phi : \mathcal{N} \times A \to A^*$ from a stackable structure, there is an *induced flow function* $\Phi : \vec{E} \to \vec{P}$, such that $\Phi(e_{g,a})$ is the path in Γ starting at the vertex g labeled by the word $\phi(y_g, a)$. Thus we have the following characterizations.

Proposition 1.2. *Let G be a group with a finite inverse-closed generating set A. (1) The group G is stackable over A if and only if the Cayley graph $\Gamma(G, A)$ admits a maximal tree with an associated bounded flow function.*
(2) The group G is autostackable over A if and only if there exists a maximal tree in $\Gamma(G, A)$ with a bounded flow function such that the graph of the induced stacking function is synchronously regular.

In Section 2 of this paper, we give definitions and notation, and discuss background on normal forms, van Kampen diagrams, and language theory.

Section 3 contains a comparison of the definitions for autostackable groups versus automatic groups. We contrast word problem solutions and van Kampen diagram constructions for these two classes of groups. In analogy with the relationship between autostackable and stackable groups above, removing the algorithmic Property (i) of Definition 3.1 of automaticity yields the definition of combable groups. In particular we show how to modify the proof of [6, Proposition 1.7] to show the following.

Proposition 3.3. *Autostackable groups are finitely presented and have solvable word problem.*

The class of automatic groups is strictly contained in the class of asynchronously automatic groups; in Section 4, we consider this larger class.

Theorem 4.1. *Every group that has an asynchronously automatic structure with a prefix-closed normal form set is autostackable.*

We note that although Epstein et. al. [10, Theorems 2.5.1,5.5.9] have shown that every automatic group has an automatic structure with respect to a set of normal forms, and also an automatic structure with respect to a prefix-closed set of not necessarily unique representatives, it is an open problem [10, Open Question 2.5.20] whether there must be an automatic structure on a prefix-closed set of normal forms. Gilman has given other characterizations of groups that are automatic with respect to a prefix-closed normal form set in [12]. Groups known to have an automatic structure with respect to prefix-closed normal forms include finite groups [10], virtually abelian (and hence Euclidean) groups and word hyperbolic groups [10], Coxeter groups [5], Artin groups of finite type [7] and of large type [28],[19], and small cancellation groups satisfying conditions $C''(p) - T(q)$ for $(p, q) \in \{(3, 6), (4, 4), (6, 3)\}$ [21]. The class of automatic groups with respect to prefix-closed normal forms is closed under graph products [15, Theorem B] and finite extensions [10, Theorem 4.1.4].

In Section 5, we give a purely algorithmic characterization of autostackability, using another type of word problem solution, namely 'prefix-sensitive rewriting'. A *convergent prefix-rewriting system* for a group G consists of a finite set A together with a subset $R \subset A^* \times A^*$ such that as a monoid, G is presented by $G = Mon\langle A \mid u = v \text{ whenever } (u, v) \in R\rangle$, and the rewriting operations of the form $uz \to vz$ for all $(u, v) \in R$ and $z \in A^*$ satisfy:

- *Normal forms:* Each $g \in G$ is represented by exactly one *irreducible* word (i.e. word that cannot be rewritten) over A.
- *Termination:* There does not exist an infinite sequence of rewritings $x \to x_1 \to x_2 \to \cdots$.

A prefix-rewriting system is *bounded* if there exists a constant k such that for each pair $(u, v) \in R$, there are words $s, t, w \in A^*$ with s and t of length at most k such that $u = ws$ and $v = wt$.

Theorem 5.3. *Let G be a finitely generated group.*
(1) The group G is stackable if and only if G admits a bounded convergent prefix-rewriting system.
(2) The group G is autostackable if and only if G admits a synchronously regular bounded convergent prefix-rewriting system.

As part of the proof of Theorem 5.3, in Proposition 5.2, we show that given any synchronously regular bounded convergent prefix-rewriting system R for G, there is a subset Q' of R that is a synchronously regular bounded prefix-rewriting system for G such that for every $(u, v) \in Q'$, every proper prefix of u is irreducible over R, and no two distinct word pairs in Q' have the same left hand side.

In contrast to these results, Otto [27, Corollary 5.3] has shown that a group is automatic with respect to a prefix-closed set of normal forms over a monoid generating set A if and only if there exists a synchronously regular convergent prefix-rewriting system such that for every $(u, v) \in R$, the word v is irreducible over R, and the word u is irreducible over all of the other rewriting rules of R.

Synchronously regular bounded convergent prefix-rewriting systems are a generalization of the more widely studied concept of finite convergent (also called complete) rewriting systems, which admit rewriting operations of the form $wuz \to wvz$ whenever $(u, v) \in R$ and $w, z \in A^*$. Thus Theorem 5.3 yields:

Corollary 5.4. *Every group that admits a finite convergent rewriting system is autostackable.*

Groups known to have a finite convergent rewriting system include finite groups, alternating knot groups [8], surface groups [23], virtually abelian groups, polycyclic groups, and more generally constructible solvable groups [13], Coxeter groups of large type [14], and Artin groups of finite type [16] (see also Le Chenadec's [24] text for many more examples). This class of groups is closed under graph products [15], extensions [13],[16], and certain amalgamated products and HNN extensions [13].

The iterated Baumslag-Solitar groups presented by $\langle a_0, a_1, ..., a_k \mid a_0^{a_1} = a_0^2, ..., a_{k-1}^{a_k} = a_{k-1}^2 \rangle$ were shown by Gersten [11, Section 6] to have Dehn function asymptotic to a k-fold iterated exponential function, and also to have a finite convergent rewriting system (see [17] for details). The following is then an immediate consequence of the results above.

Corollary 1.3. *The class of autostackable groups includes groups whose Dehn functions' growth is asymptotically equivalent to an iterated exponential function with arbitrarily many iteration steps.*

This result is in strong contrast to the quadratic upper bound on the Dehn function for any automatic group [10, Theorem 2.3.12].

Miller [26, p. 31] has shown that there exists a split extension of a finitely generated free group by another finitely generated free group that has unsolvable conjugacy problem. Since free-by-free groups admit finite complete rewriting systems, the following is immediate.

Corollary 1.4. *The class of autostackable groups includes groups with unsolvable conjugacy problem.*

Finally, we return to the motivation of computing the word problem in 3-manifold groups. In [18], Hermiller and Shapiro show that if M is a closed 3-manifold with uniform geometry that is not hyperbolic, then $\pi_1(M)$ has a finite convergent rewriting system. On the other hand, Epstein et. al. [10] show that every word hyperbolic group, and hence every hyperbolic 3-manifold fundamental group, is automatic with respect to a shortlex, and hence prefix-closed, set of normal forms. Hence we obtain the following.

Corollary 1.5. *Every fundamental group of a closed 3-manifold with uniform geometry is autostackable.*

2. Notation and background

Throughout this paper, let G be a group with a finite generating set A that is closed under inversion, and let Γ be the associated Cayley graph. Let A^* denote the free monoid, i.e. the set of all finite words over A, and let $\pi : A^* \to G$ denote the canonical surjection. Whenever u and v lie in the set A^* of all words over A, we write $u = v$ if u and v are the same word, and $u =_G v$ if u and v represent the same element of G; i.e., if $\pi(u) = \pi(v)$. Let 1 denote the identity element of G and let λ denote the empty word in A^*; then $\pi(\lambda) = 1$.

Given a word $w \in A^*$, let $l(w)$ denote the length of w as a word over A. For each $a \in A$, the symbol a^{-1} represents another element of A, and so for each word $u = a_1 \cdots a_m$ in A^* with each a_i in A, there is a formal inverse word $u^{-1} := a_m^{-1} \cdots a_1^{-1}$ in A^*.

2.1. **Normal forms and van Kampen diagrams.**

A set $\mathcal{N}$ of *normal forms* for G over A is a subset of the set A^* such that the restriction of the canonical surjection $\pi : A^* \to G$ to $\mathcal{N}$ is a bijection. As in Section 1, the symbol y_g denotes the normal form for $g \in G$; by slight abuse of notation, we use the symbol y_w to denote the normal form for $\pi(w)$ whenever $w \in A^*$.

Given a set R of defining relators for a group G, so that $\mathcal{P} = \langle A \mid R\rangle$ is a presentation for G, then for an arbitrary word w in A^* that represents the identity element 1 of G, there is a *van Kampen diagram* (or Dehn diagram) Δ for w with respect to $\mathcal{P}$. That is, Δ is a finite, planar, contractible combinatorial 2-complex with edges directed and labeled by elements of A, satisfying the properties that the boundary of Δ is an edge path labeled by the word w starting at a basepoint vertex $*$ and reading counterclockwise, and every 2-cell in Δ has boundary labeled (in some orientation) by an element of R. See [3] or [25] for more details on the theory of van Kampen diagrams.

Let $\mathcal{N}$ be a set of normal forms for G over A such that each word $w \in \mathcal{N}$ labels a simple path in the Cayley graph. For example, this property holds if $\mathcal{N}$ is closed under taking prefixes of words. The "seashell" method to construct a van Kampen diagram (with respect to the presentation $G = \langle A \mid R\rangle$) for any word $w = b_1 \cdots b_n \in A^*$ that represents the identity of G is as follows. For each i we denote the normal form word $y_i := y_{b_1 \cdots b_i}$. Let Δ_i be a van Kampen diagram for the word $y_{i-1} b_i y_i^{-1}$. By successively gluing these diagrams along the simple normal form paths along their boundaries, we obtain a planar van Kampen diagram for w; see Figure 1 for an idealized picture. (See for example [10], [2], or [6] for more details.)

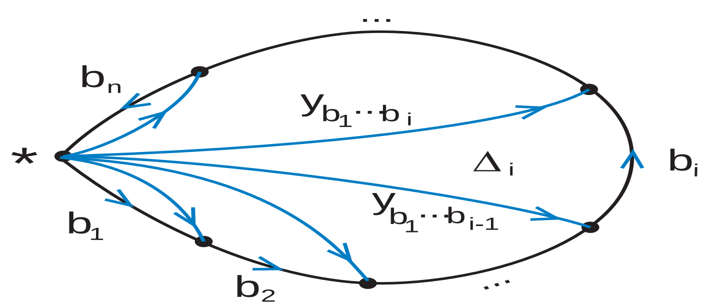

FIGURE 1. Van Kampen diagram built with seashell method

2.2. **Regular languages.** For more details and proofs of the material in this subsection, we refer the reader to [10] or [20].

A *language* over a finite set A is a subset of the set A^* of all finite words over A. We also refer to subsets of $(A^*)^n$ as languages over A^n. The set A^+ denotes the language $A^* \setminus \{\lambda\}$ of all nonempty words over A, and $A^{\leq k}$ denotes the finite language of all words over A of length at most k.

The *regular* languages over A are the subsets of A^* obtained from the finite subsets of A^* using finitely many operations from among union, intersection, complement, concatenation ($S \cdot T := \{vw \mid v \in S \text{ and } w \in T\}$), and Kleene star ($S^0 := \{\lambda\}$, $S^n := S^{n-1} \cdot S$ and $S^* := \cup_{n=0}^{\infty} S^n$). A *finite state automaton*, or FSA, is a 5-tuple $M := (A, Q, q_0, P, \delta)$, where Q is a finite set called the set of *states*, $q_0 \in Q$ is the *initial state*, $P \subseteq Q$ is the set of *accept states*, and $\delta : Q \times A \to Q$ is the *transition function*. The map δ extends to a function (often given the same label) $\delta : Q \times A^* \to Q$ by recursively defining $\delta(q, wx) := \delta(\delta(q, w), x)$ whenever $q \in Q$, $w \in A^*$, and $x \in A$. A word $w \in A^*$ is in the language accepted by M if and only if the state $\delta(q_0, w)$ lies in the set P. A language L over A is regular if and only if L is the language accepted by a finite state automaton.

The class of regular languages is closed under both image and preimage via monoid homomorphisms (see, for example, [20, Theorem 3.5]). The class of regular sets is also closed under quotients (see [20, Theorem 3.6]); we write out a special case of this in the following lemma for use in later sections of this paper.

Lemma 2.1. [20, Theorem 3.6]) *If A is a finite set, $L \subseteq A^*$ is a regular language, and $w \in A^*$, then the quotient language $L/w := \{x \in A^* \mid xw \in L\}$ is also a regular language.*

Let \$ be a symbol not contained in A. The set $A_n := (A \cup \{\$\})^n \setminus \{(\$, ..., \$)\}$ is the *padded n-tuple alphabet* derived from A. For any n-tuple of words $u = (u_1, ..., u_n) \in (A^*)^n$, write $u_i = a_{i,1} \cdots a_{i,j_i}$ with each $a_{i,m} \in A$ for $1 \leq i \leq n$ and $1 \leq m \leq j_i$. Let $M := \max\{j_1, ..., j_n\}$, and define $\tilde{u}_i := u\$^{M-j_i}$, so that each of $\tilde{u}_1$, ..., $\tilde{u}_n$ has length M. That is, $\tilde{u}_i$ is a word over the alphabet $(A \cup \{\$\})^*$, and we can write $\tilde{u}_i = c_{i,1} \cdots c_{i,M}$ with each $c_{i,m} \in A \cup \{\$\}$. The word $\mu(u) := (c_{1,1}, ..., c_{n,1}) \cdots (c_{1,M}, ..., c_{n,M})$ is the *padded word* over the alphabet A_n induced by the n-tuple $(u_1, ..., u_n)$ in $(A^*)^n$.

A subset $L \subseteq (A^*)^n$ is called *synchronously regular* if the *padded extension* set $\mu(L) := \{\mu(u) \mid u \in L\}$ of padded words associated to the elements of L is a regular language over the alphabet A_n. The class of synchronously regular languages is closed under finite unions and intersections, since the padded extension of a union [resp. intersection] is the union [resp. intersection] of the padded extensions. We also include two lemmas on synchronously regular languages for use in later sections. The first lemma says that the "diagonal" of a regular set is regular.

Lemma 2.2. *If L is a regular language over an alphabet A, then the set $\Delta(L) := \{\mu(w, w) \mid w \in L\}$ is a regular language over the alphabet $A_2 = (A \cup \$)^2 \setminus \{(\$, \$)\}$.*

Proof. Given an expression of the regular language L using letters of A together with the operations $\cup, \cap, (\ )^c, \cdot, (\ )^*$, replace every instance of a letter $a \in A$ with the letter $(a, a) \in A_2$. □

Lemma 2.3. *If $L_1, ..., L_n$ are regular languages over A, then their Cartesian product $L_1 \times \cdots \times L_n \subseteq (A^*)^n$ is synchronously regular.*

Proof. For each $1 \leq i \leq n$ define the monoid homomorphism $\rho_i : A_n^* \to (A \cup \$)^*$ by $\rho_i(a_1, ..., a_n) := a_i$. Then the padded extension of the product language $L := L_1 \times \cdots \times L_n$ satisfies $\mu(L) = \cap_{i=1}^n \rho_i^{-1}(L_i \$^*)$. Since each language $L_i\$^*$ is regular, and regular languages are closed under homomorphic preimage and finite intersection, then $\mu(L)$ is regular. □

A (deterministic) *asynchronous (two tape) automaton* over A is a finite state automaton $M = (A \cup \{\#\}, Q, q_0, P, \delta)$ satisfying: (1) The state set Q is a disjoint union $Q = Q_1 \cup Q_1^\# \cup Q_2 \cup Q_2^\# \cup \{q_f\} \cup \{F\}$ of six subsets, the initial state q_0 lies in $Q_1 \cup Q_2$, and the set of accept states is $P = \{q_f\}$. (2) The transition function $\delta : Q \times (A \cup \{\#\}) \to Q$ satisfies $\delta(q, a) \in Q_1 \cup Q_2 \cup \{F\}$ if $q \in Q_1 \cup Q_2$ and $a \in A$; $\delta(q, a) \in Q_1^\# \cup \{F\}$ if either ($q \in Q_2$ and $a = \#$) or ($q \in Q_1^\#$ and $a \in A$); $\delta(q, a) \in Q_2^\# \cup \{F\}$ if either ($q \in Q_1$ and $a = \#$) or ($q \in Q_2^\#$ and $a \in A$); $\delta(q, a) \in \{q_f, F\}$ if $q \in Q_1^\# \cup Q_2^\#$ and $a = \#$; and $\delta(q, a) = F$ if $q = F$ and $a \in A \cup \{\#\}$. As before, extend δ to a function $\delta : Q \times (A \cup \{\#\})^* \to Q$ recursively by $\delta(q, wa) := \delta(\delta(q, w), a)$.

This finite state automaton is viewed as reading from two tapes rather than one, by the interpretation that the words on each tape are to have an ending symbol # appended, and when the automaton M is in a state in $Q_i \cup Q_i^\#$, then M will read the next symbol from tape i. Then the automaton is in a state of $Q_i^\#$ after M has finished reading the word on the other tape.

More precisely, given a pair of words $(u,v) \in ((A \cup \{\#\})^*)^2$, a *shuffle* of (u,v) is a word $u_1v_1 \cdots u_jv_j \in (A \cup \{\#\})^*$ such that each $u_i, v_i \in (A \cup \{\#\})^*$, $u = u_1 \cdots u_j$, and $v = v_1 \cdots v_j$. Let $(u,v) \in ((A \cup \{\#\})^*)^2$, and write $u = a_{1,1} \cdots a_{1,m_1}$ and $v = a_{2,1} \cdots a_{2,m_2}$ where each $a_{i,j} \in A \cup \{\#\}$. Given a state $q \in Q$ and the pair (u,v), there is a unique word $\sigma_{M,q}(u,v) := c_1 \cdots c_{m_1+m_2} \in (A \cup \{\#, F\})^*$ defined recursively, such that $c_1 := a_{i,1}$ if $q \in Q_i \cup Q_i^{\#}$ (and $1 \leq m_i$) and $c_1 := F$ if $q \in \{q_f, F\}$, and whenever $k \leq m_1 + m_2 - 1$, if $c_1 \cdots c_k$ is a shuffle of $(a_{1,1} \cdots a_{1,k_1}, a_{2,1} \cdots a_{2,k_2})$ with $\delta(q, c_1 \cdots c_k) = q'$, then $c_{k+1} := a_{i,k_i+1}$ if $q' \in Q_i \cup Q_i^{\#}$ (and $k_i < m_i$) and $c_{k+1} := F$ if $q' \in \{q_f, F\}$; and if $c_k = F$ then $c_{k+1} := F$.

A pair $(u,v) \in (A^*)^2$ is accepted by the asynchronous automaton M if and only if $\sigma_{M,q_0}(u\#, v\#)$ is a shuffle of $(u\#, v\#)$; i.e., there is no occurrence of the letter F. (Equivalently, (u_1, u_2) is in the language of M if and only if the machine M reads the next letter from the $u_i\#$ tape whenever M is in a state of $Q_i \cup Q_i^{\#}$, M starts in state q_0, and M ends in state q_f when both tapes have been read.) A subset of $A^* \times A^*$ is an *asynchronously regular* language if it is the set of word pairs accepted by an asynchronous automaton.

Again we include a closure property for asynchronously regular languages for later use. This result is proved by Rabin and Scott in [29, Theorem 16].

Lemma 2.4. [29] *If $L \subset (A^*)^2$ is an asynchronously regular language, then the projection on the first coordinate given by the set $\rho_1(L) := \{u \mid \exists (u,v) \in L\}$ is a regular language over A.*

3. Autostackable versus automatic: Word problems and van Kampen diagrams

We give a definition of automatic structures for groups that is equivalent to, but differs from, the original definition in [10], in order to illustrate more completely the close connection to Definition 1.1 of autostackable structures above. Both automaticity and autostackability utilize the concepts of a set $\mathcal{N}$ of normal forms for a group G over a generating set A, but in contrast to the stacking function ϕ for autostackability which has a finite image set, the definition of automaticity relies on the *normal form map* $\mathrm{nf}_{\mathcal{N}} : \mathcal{N} \times A \to A^*$ defined by $\mathrm{nf}_{\mathcal{N}}(y_g, a) := y_{ga}$.

Definition 3.1. *A group G with finite inverse-closed generating set A is* automatic *if there are a set $\mathcal{N}$ of normal forms for G over A and a constant k such that the following hold:*

(i) *The graph of the function $\mathrm{nf}_{\mathcal{N}} : \mathcal{N} \times A \to A^*$,*

$$graph(\mathrm{nf}_{\mathcal{N}}) := \{(y_g, a, y_{ga}) \mid g \in G, a \in A\},$$

is a synchronously regular language.

(ii) *For each $g \in G$ and $a \in A$, the pair of paths in Γ labeled y_g and y_{ga} beginning at the identity vertex 1 and ending at the endpoints of $e_{g,a}$*

must k-fellow travel*; that is, for any natural number* i*, if* w *and* w' *are the length* i *prefixes of the words* y_g *and* y_{ga}*, then there must be a path in* Γ *of length at most* k *between the vertices of* Γ *labeled by* w *and* w'*.*

In fact, the definition of automaticity given in [10, Defn. 2.3.1,Thm. 2.5.1] requires only property (i) above; indeed, in [10, Thm. 2.3.5,Thm. 3.3.4] Epstein et. al. show that the geometric property (ii) follows from the algorithmic property (i). Moreover, it is immediate from the properties of regular languages discussed in Section 2.2 that the set $graph(\mathrm{nf}_{\mathcal{N}})$ is a synchronously regular language if and only if the sets $L_a := \{(y_g, y_{ga}) \mid g \in G\} \subset (A^*)^2$ are synchronously regular for each $a \in A \cup \{\lambda\}$, giving the equivalence of property (i) above with the definition in [10].

Comparing Definitions 1.1 and 3.1, the automatic property (i) requires a finite state automaton that can recognize the tuple (y_g, a, z) where the third coordinate is the normal form $z = y_{ga}$, but the autostackable property (1) requires only a FSA that recognizes such a tuple in which z is a bounded length word giving information toward eventually finding the normal form y_{ga}. (We make this more precise below.) In analogy with the autostackable property (2) of Definition 1.1, the automatic group property (ii) naturally divides into degenerate and recursive cases, in that if the directed edge $e_{g,a}$ is degenerate, we have the stronger property that the paths y_g, y_{ga} 1-fellow travel.

Analogous to the relationship between autostackable and stackable groups, removing the algorithmic property (i), a group G is called *combable* over A if the geometric property (ii) of Definition 3.1 holds for some set $\mathcal{N}$ of normal forms and some constant k. Note that combability, and hence also automaticity, imply finite presentability; in particular, the set R of all words of length up to $2k+2$ that represent the identity are a set of defining relators for the group.

If G is a combable group satisfying the further property that the words of the normal form set $\mathcal{N}$ label simple paths in the Cayley graph Γ, for example in the case that $\mathcal{N}$ is closed under taking prefixes, then the "seashell" method discussed in Section 2.1 extends to the following procedure to construct a van Kampen diagram (with respect to the presentation induced by the combable structure) for any word that represents the identity of G. Given a word $w = b_1 \cdots b_n$ representing the identity of G, with each $b_i \in A$, let $y_i := y_{b_1 \cdots b_i}$ for each i. Property (ii) shows that for each i there is a van Kampen diagram Δ_i labeled by $y_{i-1} b_i y_i^{-1}$ that is "k-thin" as illustrated in Figure 2. Gluing these k-thin diagrams along their y_i boundaries results in a planar van Kampen diagram for w; see Figure 1. In the case that the group is automatic, this yields a solution of the word problem. (See [10] for full details.)

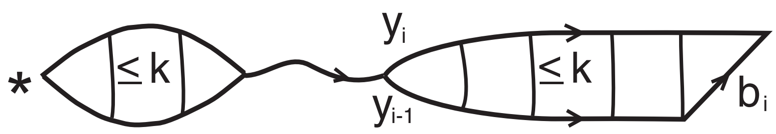

FIGURE 2. "k-thin" van Kampen diagram

In [6], the first two authors of this paper show that every stackable group G also has a finite presentation and admits a procedure for building van Kampen diagrams. Moreover, in the case that the set $S_\phi := \{(w, a, \phi(e_{\pi(w),a})) \mid w \in A^*, a \in A\}$ obtained from the stacking function ϕ is decidable, they show that the procedure is an effective algorithm and the group has solvable word problem. In part (1) of Definition 1.1 above, however, the synchronously regular (and hence recursive) set $graph(\phi)$ is a subset of S_ϕ, namely $graph(\phi) = S_\phi \cap (\mathcal{N} \times A \times A^*)$. We alter the stacking reduction procedure of [6] to solve the word problem for autostackable groups as follows.

For a group G with a stackable structure given by a set $\mathcal{N}$ of normal forms over a inverse-closed generating set A and a stacking function $\phi : \mathcal{N} \times A \to A^*$, the *stacking reduction algorithm* on words over A is a prefix-rewriting system given by

$$\begin{aligned} R_\phi :=& \{(ya, y\phi(y,a)) \mid y \in \mathcal{N}, a \in A, ya \notin \mathcal{N} \cup A^*a^{-1}a\} \\ &\cup \{(yaa^{-1}, y) \mid ya \in \mathcal{N}, a \in A\}. \end{aligned}$$

Recall that starting from any word w in A^*, whenever we can decompose w as $w = ux$ for some rule $(u, v) \in R_\phi$ and word $x \in A^*$, then we can rewrite $w \to vx$. Each of these rewritings consists either of free reduction or *ϕ-reduction*.

Lemma 3.2. *If G is a group with inverse-closed generating set A and a stackable structure consisting of a normal form set $\mathcal{N}$ and a stacking function ϕ, then the prefix-rewriting system R_ϕ is a convergent prefix-rewriting system for G.*

Proof. Let w be any word in A^*, and write $w = b_1 \cdots b_m$ with each b_i in A. Suppose that $w' = c_1 \cdots c_n$, with each c_j in A, is obtained from w by repeated applications of free and ϕ-reductions, and that $w' \to w''$ is a single instance of another R_ϕ rewriting operation. If the rewriting $w' \to w''$ is a free reduction, then two letters of w' are removed, and if this rewriting is a ϕ-reduction, then a single letter of w' is replaced by a bounded length word. Inductively this shows that each letter of the word w'' is the result of successive rewritings from a specific letter b_i of the original word w. Viewing this topologically, if the rewriting operation $w' \to w''$ is free reduction, then the directed path in the Cayley graph $\Gamma(G, A)$ starting at 1 and labeled w'' is obtained from the path labeled by w' via the removal of two edges, and if the rewriting is ϕ-reduction, then a single edge e' of the w' path is replaced by the path $\Phi(e')$, where Φ is the flow function induced by the stacking function ϕ. In the latter case, there is a specific recursive edge $e_i := e_{\pi(b_1 \cdots b_{i-1}),b_i} \in \vec{E}_{\mathcal{N},r}$ for some index $1 \leq i \leq m$ on the path labeled w from 1 in Γ such that e' was obtained from e_i via successive applications of the flow function, and for each recursive edge e'' along the path $\Phi(e')$, we have $e'' <_\phi e'$, where $<_\phi$ is the strict well-founded partial ordering given in Definition 1.1(2r). Since at each application of the flow function a bounded number of recursive edges are added to the path, König's Infinity Lemma (see, for example, [9, Lemma 8.1.2]) shows that at most finitely many Φ-reductions can be applied starting from each of the finitely many edges of the original path labeled w. Hence only finitely many ϕ-reductions can be applied in any sequence of rewritings starting from the word w. Between these ϕ-reductions, only finitely many free reductions can occur. Hence after finitely many R_ϕ rewriting operations, we must obtain an irreducible word y_w, and so the prefix-rewriting system R_ϕ is terminating.

Now suppose that y is any irreducible word with respect to R_ϕ. Write $y = a_1 \cdots a_n$ with each a_i in A and $y_i := a_1 \cdots a_i$ for each i, and suppose that y_j is the shortest prefix of y that does not lie in $\mathcal{N}$. Since the empty word λ lies in the normal form set $\mathcal{N}$ of the stackable structure, we have $j \geq 1$. Now $y_{j-1} \in \mathcal{N}$, and either $y_{j-1} = y_{j-2}a_j^{-1}$, in which case a free reduction rule of R_ϕ applies to y, or else y_{j-1} does not end with the letter a_j^{-1}, in which case a ϕ-reduction rule applies to y. However, this contradicts the irreducibility of y. Therefore every prefix of the word y, including the word y itself, must lie in $\mathcal{N}$. Thus the set of irreducible words with respect to R_ϕ is contained in the set $\mathcal{N}$ of normal forms for G.

Next suppose that w is any word in the normal form set $\mathcal{N}$. By the termination proof above, there is a finite sequence of rewritings from w to an irreducible word y_w. Since every pair of words in the prefix-rewriting system R_ϕ represents the same element of the group G, then $w =_G y_w$. By the previous paragraph, the irreducible word y_w must lie in $\mathcal{N}$. But since each element of G has exactly one representative in $\mathcal{N}$, this implies that $w = y_w$. Hence the set $\mathcal{N}$ of normal forms for the stackable structure is equal to the set of irreducible words with respect to the prefix-rewriting system R_ϕ. Note that this shows both that the set $\mathcal{N}$ is prefix-closed, and that the set of R_ϕ-irreducible words are a set of normal forms. Hence R_ϕ is convergent.

Finally, since A is a monoid generating set for G, and the rules of R_ϕ define relations of G that give a set of normal forms for G, the convergent prefix-rewriting system R_ϕ gives a monoid presentation of G. □

Recall that the normal form of the identity element in an autostackable group must be the empty word. Decidability of the set *graph*(ϕ) implies that for any word $w \in A^*$, one can determine whether or not a ϕ-reduction applies, and so Lemma 3.2 completes the word problem solution in that case.

Another immediate consequence of Lemma 3.2 is that the *stacking presentation*

$$G = \langle A \mid \{\phi(y_g, a)a^{-1} \mid g \in G, a \in A\}\rangle$$

is a (group) presentation for the stackable group G. Property (2) of the definition of stackable implies that this presentation is finite. Hence we have the following.

Proposition 3.3. *Autostackable groups are finitely presented and have solvable word problem.*

In [6, Proposition 1.12], the first two authors of this paper show how to use computability of the set S_ϕ to obtain an inductive algorithm for constructing van Kampen diagrams over this presentation. A similar alteration of the proof shows that this algorithm applies in the case that *graph*(ϕ) is recursive. However, in [6, Proposition 1.12], another hypothesis was included, that the generating set A of the stackable structure did not include a letter representing the identity element of the group. We note that given any autostackable structure for a group G, with inverse-closed generating set A, normal forms $\mathcal{N}$, and stacking function ϕ, if $A' \subset A$ is the set of letters in A representing 1, then since the normal form set is prefix-closed, no element of $\mathcal{N}$ can contain a letter from A'. It can then be shown that G is also autostackable over the inverse-closed generating set $B := A \setminus A'$, with the same normal form set $\mathcal{N}$, and the stacking function $\phi' : \mathcal{N} \times B \to B^*$ given by setting $\phi'(y, b)$ equal to the word $\phi(y, b)$ with all instances of letters in A' removed.

We include a few more details of this inductive algorithm here to illustrate the difference between the van Kampen diagrams built from an autostackable structure and those built from a prefix-closed automatic structure. For an autostackable group, since the set of normal forms is prefix-closed, each normal form word must label a simple path in the Cayley graph Γ, and as in the case of automatic groups, we extend the "seashell" method described in Section 2.1 to a diagram-building algorithm. Given a word $w = b_1 \cdots b_n$ with each $b_i \in A$ and such that $\pi(w) = 1$, and letting $y_i := y_{b_1 \cdots b_i}$ for each i, this method requires an algorithm for building van Kampen diagrams Δ_i for the words $y_{i-1}b_iy_i^{-1}$, which then can be glued as in Figure 1 to obtain the diagram for w. However, in this case the van Kampen diagram Δ_i will not be "thin" in general, but instead is built by recursion using property (2) of Definition 1.1. If the directed edge e_{y_{i-1},b_i} of Γ is degenerate, then the van Kampen diagram Δ_i is homeomorphic to a line segment, containing no 2-cells; this is pictured in Figure 3. On the other hand, if the edge e_{y_{i-1},b_i} is recursive, and we write $\phi(y_{i-1}, b_i) = a_1 \cdots a_m$ with each $a_j \in A$, then by Noetherian induction (using the well-founded strict partial ordering $<_\phi$) we may assume that for each $1 \le j \le m$ we have already built

FIGURE 3. Degenerate van Kampen diagrams

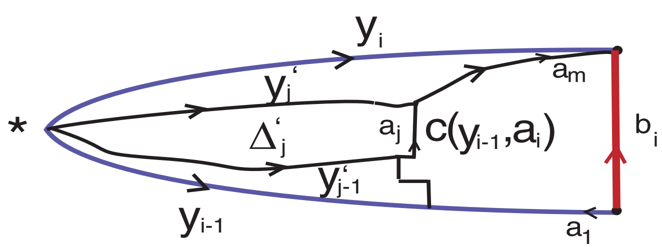

FIGURE 4. Recursive van Kampen diagram

a van Kampen diagram Δ'_j for the word $y'_{j-1}a_jy'^{-1}_j$, where y'_j denotes the normal form word representing the element $y_{i-1}a_1\cdots a_j$ for each j. Successively gluing these diagrams Δ'_j, or *stacking* them, along their common (simple) boundary paths y'_j, we obtain a planar diagram with boundary word $y_{i-1}\phi(y_{i-1},b_i)y_i^{-1}$. Finally, glue on a single 2-cell whose boundary is labeled by the word $\phi(y_{i-1},b_i)^{-1}b_i$ to obtain the required van Kampen diagram Δ_i. This process is illustrated in Figure 4.

4. Asynchronously automatic groups

A group G with finite inverse-closed generating set A is asynchronously automatic if there is a regular language $\mathcal{N} = \{y_g \mid g \in G\}$ of normal forms for G over A such that for each $a \in A$ the subset

$$L_a := \{(y_g, y_{ga}) \mid g \in G\}$$

of $A^* \times A^*$ is an asynchronously regular language. Every automatic group is also asynchronously automatic. This section is devoted to the proof of Theorem 4.1.

Theorem 4.1. *Every group that has an asynchronously automatic structure with a prefix-closed normal form set is autostackable.*

Proof. Let G be an asynchronously automatic group with finite inverse-closed generating set A and prefix-closed normal form set $\mathcal{N}$, and for each $a \in A$ let $M_a = (A \cup \{\#\}, Q_a, q_0, P_a, \delta_a)$ be an asynchronous automaton accepting the language L_a. By [10, Theorem 7.2.4], we may also assume that the asynchronously automatic structure is *bounded*. That is, there is a constant C such that for each pair $(u, v) \in L_a$, the shuffle $\sigma_{M_a,q_0}(u\#, v\#)$

(see Section 2.2 for the definition of this notation) of the pair $(u\#, v\#)$ has the form $\sigma_{M_a,q_0}(u,v) = u_1v_1 \cdots u_mv_m$ where $u = u_1 \cdots u_m$, $v = v_1 \cdots v_m$, each $u_i, v_i \in (A \cup \{\#\})^*$, and the lengths of these subwords satisfy $0 \leq l(u_1) \leq C$, $1 \leq l(u_i) \leq C$ for all $2 \leq i \leq m$, $1 \leq l(v_i) \leq C$ for all $1 \leq i \leq m-1$, and $0 \leq l(v_m) \leq C$.

By increasing this constant if necessary, we may also assume that C is greater than 3 and greater than the cardinality $|Q_a|$ of the set of states of the automaton M_a for every $a \in A$. We can view M_a as a finite graph with vertex set Q_a and a directed edge labeled $b \in A \cup \{\#\}$ from $\hat{q}$ to $\tilde{q}$ whenever $\delta_a(\hat{q}, b) = \tilde{q}$. Let Q_a^{good} be the set of all states q in $Q_{a,1} \cup Q_{a,2}$ such that there is a path in M_a from the initial state q_0 to q and there also is a path in M_a from q to the accept state q_f. For each $q \in Q_a^{good}$, by eliminating repetition of vertices along the path to q_f, there must also be a directed edge path in M_a from q to q_f of length less than C. Let $W_q \in (A \cup \{\#\})^*$ be a fixed choice of such a word, for each such q. Note that this word must contain two instances of the letter $\#$, and we can write $W_q = \sigma_{M_a,q}(p_q\#, r_q\#)$ for two words $p_q, r_q \in A^*$ satisfying $l(p_q) + l(r_q) \leq C - 2$.

To define the function $\phi : \mathcal{N} \times A \to A^*$, we first set $\phi(y_g, a) := a$ whenever the edge $e_{g,a}$ lies in the set $\vec{E}_d = \vec{E}_{\mathcal{N},d}$ of degenerate edges of the Cayley graph $\Gamma(G, A)$ with respect to the set $\mathcal{N}$ of normal forms; i.e. whenever either $y_g a = y_{ga}$ or $y_{ga}a^{-1} = y_g$, as required for property (2d) of Definition 1.1. Now suppose that $e_{g,a}$ is recursive. If $l(y_g) + l(y_{ga}) \leq C^2 + 3C$, then define $\phi(y_g, a) := y_g^{-1} y_{ga}$.

On the other hand, suppose that $l(y_g) + l(y_{ga}) > C^2 + 3C$. The pair (y_g, y_{ga}) is accepted by the asynchronous automaton M_a, and so the word $w := \sigma_{M_a,q_0}(y_g\#, y_{ga}\#)$ is a shuffle of $(y_g\#, y_{ga}\#)$ and satisfies $\delta_a(q_0, w) = q_f$, the accept state of M_a. The bounded property above implies that $l(y_{ga}\#) \leq Cl(y_g\#)$, and so $l(y_g\#) + Cl(y_g\#) > C^2 + 3C + 2$, which gives $l(y_g) > C + 1$. Write $w = w'w''$ where w'' is the shortest suffix of w containing exactly $C + 1$ letters from y_g (i.e., $C + 2$ letters from $y_g\#$). Applying the bounded property again shows that $w' = \sigma_{M_a,q_0}(u, v)$ and $w'' = \sigma_{M_a,q}(s\#, t\#)$ where $q = \delta_a(q_0, w') \in Q_a^{good}$, $u, v, s, t \in A^*$, $us = y_g$, $vt = y_{ga}$, $l(s) = C + 1$, and $1 \leq l(t\#) \leq C^2 + 2C$. Returning to the view of M_a as a finite graph, the word w' labels a path from q_0 to q and w'' labels a path from q to q_f. Thus the word $\tilde{w} := w'W_q$ also labels a path from q_0 to q_f, and so the pair (up_q, vr_q) lies in the language L_a. Note that $usa =_G vt$ and $up_q a =_G vr_q$, and so $s^{-1}p_q a r_q^{-1} t =_G a$. Moreover, the pair (s, t) and the state q are uniquely determined by (y_g, y_{ga}); i.e., by g and a. In this case we define $\phi(y_g, a) := s^{-1} p_q a r_q^{-1} t$.

With this definition of the stacking function ϕ, the length of the word $\phi(y_g, a)$ is at most $C^2 + 4C - 1$ for all $g \in G$ and $a \in A$, and in each case $\phi(y_g, a) =_G a$.

Now suppose that $e = e_{g,a}$ and $e' = e_{g',a'}$ are recursive edges such that e' lies on the directed path in the Cayley graph Γ starting at the vertex g and labeled by the word $\phi(y_g, a)$. Now when $l(y_g) + l(y_{ga}) \leq C^2 + 3C$ the path $\phi(y_g, a) = y_g^{-1}y_{ga}$ follows only degenerate edges, so we must have $l(y_g) + l(y_{ga}) > C^2 + 3C$. In this case, the path starting at g and labeled by the word $\phi(y_g, a) := s^{-1}p_q a r_q^{-1} t$ defined above follows only degenerate edges along the subpaths labeled by $s^{-1}p_q$ and $r_q^{-1}t$, since $us = y_g$, $vt = y_{ga}$, and $(up_q, vr_q) \in L_a$, and so the words $vr_q, vt \in \mathcal{N}$ as well. So the edge e' must be the edge labeled $a' = a$ with initial vertex $g' =_G gs^{-1}p_q =_G up_q$. That is, we have normal forms $y_g = us$ with $l(s) = C + 1$ and $y_{g'} = up_q$ with $l(p_q) \leq C$, and so the normal form to the initial vertex of e' is strictly shorter than the normal form to the initial vertex of e. Hence the relation $<_\phi$ defined in property (2r) of Definition 1.1 strictly increases the length of the normal form of the initial vertex of the edges, and so is a well-founded strict partial ordering. Therefore G is stackable over A.

By hypothesis the normal form set $\mathcal{N}$ is a regular language, and so for each $a \in A$, an application of Lemma 2.1 shows that the language $J_a := \{y \mid ya \in \mathcal{N}\}$ is regular. Lemma 2.3 then shows that the languages $J_a \times \{a\} \times \{a\}$ and $(\mathcal{N} \cap A^* a^{-1}) \times \{a\} \times \{a\}$ are synchronously regular. The finite union of these sets for $a \in A$ is the subset of $graph(\phi)$ corresponding to the application of ϕ to degenerate edges, and therefore this set is synchronously regular.

The subset

$$L_{smallrec} := \{(y_g, a, \phi(y_g, a)) \mid g \in G, a \in A, e_{g,a} \in \vec{E}_r \text{ and } l(y_g)+l(y_{ga}) \leq C^2+3C\}$$

of $graph(\phi)$ is finite, and therefore also is synchronously regular. For use in avoiding overlapping sets later, denote $J_{a,smallrec} := \{y_g \mid (y_g, a, \phi(y_g, a)) \in L_{smallrec}\}$.

For each $a \in A$ and $q \in Q_a^{good}$, let

$$K_{a,q} := \{(u, v) \mid u, v \in A^*, \delta_a(q_0, \sigma_{M_a,q_0}(u, v)) = q\},$$

and note that by definition of Q_a^{good} the set $K_{a,q}$ is nonempty. This subset of $A^* \times A^*$ is asynchronously regular; in particular, if $q \in Q_{a,1}$, then $K_{a,q}$ is the accepted language of the asynchronous automaton $\widetilde{M} = (A \cup \{\#\}, \widetilde{Q}, q_0, P_a, \tilde{\delta})$ where $\widetilde{Q}_1 = Q_{a,1}$, $\widetilde{Q}_2 = Q_{a,2}$, $\widetilde{Q}_1^{\#} = \emptyset$, $\widetilde{Q}_2^{\#} = \{\tilde{q}\}$, and $\tilde{\delta}(q', b) = \delta(q', b)$ for all $q' \in \widetilde{Q}_{a,1} \cup \widetilde{Q}_{a,2}$ and $b \in A$, $\tilde{\delta}(q, \#) = \tilde{q}$, $\tilde{\delta}(\tilde{q}, \#) = q_f$, and $\tilde{\delta}(q', b) = F$ otherwise. The case that $q \in Q_{a,2}$ is similar. Then Lemma 2.4 shows that the set $\rho_1(K_{a,q}) = \{u \mid \exists (u, v) \in K_{a,q}\}$ is a regular language.

Let

$$S_{a,q} := \{(s, t) \mid s, t \in A^*, l(s) = C + 1, \text{ and } \delta(q, \sigma_{M_a,q}(s\#, t\#)) = q_f\}.$$

The boundedness of the asynchronously automatic structure implies that $l(t) < C^2 + 2C$, and the set $S_{a,q}$ is finite. Moreover, note that if $(s, t), (s, t') \in$

$S_{a,q}$, then if we let (u, v) be an element of the nonempty set $K_{a,q}$, we have $(us, vt), (us, vt') \in L_a$, and so vt, vt' are both normal form words representing the same element $\pi(usa)$ of G; hence $t = t'$. Thus for each $s \in A^{C+1}$, there is at most one word $t_{q,s}$ such that the pair $(s, t_{q,s}) \in S_{a,q}$.

Next for each $a \in A$, $q \in Q_a^{good}$, and $(s, t) \in S_{a,q}$, let

$$L_{a,q,s} := \rho_1(K_{a,q})s \cap [A^* \setminus (J_a \cup (\mathcal{N} \cap A^*a^{-1}) \cup J_{a,smallrec})].$$

This is the set of words $us \in A^*s$ such that $us \in \mathcal{N}$, the edge $e_{\pi(us),a}$ is recursive, $l(us) + l(y_{usa}) > C^2 + 3C$, and the path labeled $\sigma_{M_a,q_0}(us, y_{usa})$ goes from q_0 through q to q_f in M_a. Closure properties of regular sets shows that this language is regular. Applying Lemma 2.3 again, the language $L_{a,q,s} \times \{a\} \times \{s^{-1}p_q a r_q^{-1} t_{q,s}\}$ is a synchronously regular subset of $graph(\phi)$ corresponding to these recursive edges.

We can now write the graph of the stacking function ϕ as the finite union

$$\begin{aligned} graph(\phi) = {} & [\cup_{a \in A}(J_a \cup (\mathcal{N} \cap A^*a^{-1})) \times \{a\} \times \{a\})] \\ & \cup L_{smallrec} \\ & \cup [\cup_{a \in A, q \in Q_a^{good}, (s, t_{q,s}) \in S_{a,q}} (L_{a,q,s} \times \{a\} \times \{s^{-1}p_q a r_q^{-1} t_{q,s}\})]. \end{aligned}$$

Closure of the class of synchronously regular sets under finite unions then shows that $graph(\phi)$ is synchronously regular. Thus G is autostackable. □

5. Rewriting systems

In this section we prove the characterization of autostackable groups in terms of synchronously regular bounded convergent prefix-rewriting systems, and conclude with a discussion of finite convergent rewriting systems. We begin by discussing a process for minimizing prefix-rewriting systems.

Definition 5.1. *A convergent prefix-rewriting system $R \subset A^* \times A^*$ for a group G is* processed *if:*

(a) *For each $a \in A$ there is a letter in A, which we denote a^{-1}, such that $\pi(a)^{-1} = \pi(a^{-1})$ (where $\pi : A^* \to G$ is the canonical map).*
(b) *For each pair $(u, v) \in R$, every proper prefix of u is irreducible with respect to the rewriting operations of R.*
(c) *Whenever $(u, v_1), (u, v_2) \in R$, then $v_1 = v_2$.*

For any prefix-rewriting system R over A, let $\mathrm{Irr}(R)$ denote the set of irreducible words with respect to the rewriting operations $ux \to vx$ whenever $(u, v) \in R$ and $x \in A^*$. Note that every prefix of a word in $\mathrm{Irr}(R)$ must also lie in $\mathrm{Irr}(R)$.

Proposition 5.2. *If a group G admits a synchronously regular bounded convergent prefix-rewriting system R over a monoid generating set B, then*

G also admits a processed synchronously regular bounded convergent prefix-rewriting system Q over the generating set $A := B \cup B^{-1}$, such that Irr(R) = Irr(Q).

Proof. Let R be a bounded convergent prefix-rewriting system over B for G, and let $\pi : B^* \to G$ be the associated surjective monoid homomorphism. For each element $b \in B$, let the symbol b^{-1} denote another letter, and let $A := B \cup \{b^{-1} \mid b \in B\}$. For each $b \in B$, let z_b denote the unique word in $\text{Irr}(R)$ representing the element $\pi(b^{-1})$ of G.

Let
$$R' := \{(yb, v) \mid (yb, v) \in R, y \in \text{Irr}(R), b \in B\};$$
i.e., the set of all rules of R whose left entry has every proper prefix irreducible; i.e., that satisfies property (b) of Definition 5.1.

Let k be the constant associated to the bounded property of the prefix-rewriting system R. Then by expressing the finite set
$$W := \{(s,t) \in B^{\leq k} \times B^{\leq k} \mid \text{the first letters of } s \text{ and } t \text{ are distinct}\},$$
as $W = \{(s_1, t_1), ..., (s_n, t_n)\}$, we can write each element $r := (u, v)$ of R in the form $r = (ws_{i(r)}, wt_{i(r)})$ for a unique index $i(r) \in \{1, ..., n\}$ and word $w \in B^*$. For each $1 \leq i \leq n$, let
$$R'_i := \{r \in R' \mid i(r) = i\}.$$
Then R' is the disjoint union $R' = \cup_{i=1}^n R'_i$. Note that if there are two pairs $r_1 = (u, v_1), r_2 = (u, v_2) \in R'$ that have the same left hand entry but $v_1 \neq v_2$ on the right, then the indices $i(r_1) \neq i(r_2)$ must also be distinct. Let
$$Q' := \{r = (u, v) \in R' \mid \nexists \tilde{r} = (u, \tilde{v}) \in R' \text{ with } i(\tilde{r}) < i(r)\}.$$
That is, the subset Q' of R satisfies properties (b) and (c) of Definition 5.1. We then define the prefix-rewriting system $Q := Q' \cup Q''$ where
$$Q'' := \{(yb^{-1}, yz_b) \mid y \in \text{Irr}(R) \setminus B^*b\} \cup \{(ybb^{-1}, y) \mid yb \in \text{Irr}(R), b \in B\}.$$

Suppose that $w \in A^*$ is rewritten by a sequence of applications of rewriting operations using the prefix-rewriting system Q. Since the only occurrences of letters of B^{-1} in Q appear in left hand sides of pairs in Q'', at most $l(w)$ of the rewritings in this sequence involve a rule of Q''. The rules in Q' all lie in the convergent prefix-rewriting system R, which satisfies the termination property, and so only finite sequences of applications of Q' rules can occur. Hence there can be at most finitely many rewritings in any such rewriting of w; that is, the prefix-rewriting system Q is terminating.

Suppose that w is any word in $\text{Irr}(R)$. Then $w \in B^*$, so w can't be rewritten using a pair from Q'', and since $Q' \subseteq R$, the word w also can't be reduced using Q'. Hence $\text{Irr}(R) \subseteq \text{Irr}(Q)$.

On the other hand, suppose that x is a word in $\text{Irr}(Q) \setminus \text{Irr}(R)$. If $x \in B^*$, then write $x = x'bx''$ where $x' \in B^*$, $b \in B$, and $x'b$ is the shortest prefix of x that does not lie in $\mathrm{Irr}(R)$. But then there must be a pair $(u, v) \in R$ and a word $z \in B^*$ such that $x'b = uz$. Since x' is irreducible over R, then $x'b = u$ and $z = \lambda$, and the rule (u, v) also lies in the subset R' of R. Hence there also is a rule (u, v') in the subset Q' of Q, since the sets of left hand sides of rules of R' and Q' are the same. But then x is reducible over Q. This contradiction implies that $B^* \cap \mathrm{Irr}(Q) \subseteq \mathrm{Irr}(R)$, and so we must have $x \notin B^*$. In this case we can write $x = x'ax''$ with $a \in B^{-1}$ and $x' \in B^*$, where a is the first occurrence of a letter of $A \setminus B$ in x. Since the set of irreducible words over a prefix-rewriting system is prefix-closed, the word x' lies in $\mathrm{Irr}(Q) \cap B^*$, and hence also in $\mathrm{Irr}(R)$. But then the word x can be reduced using an element of Q'', another contradiction. Therefore we have $\mathrm{Irr}(R) = \mathrm{Irr}(Q)$. Since the set $\mathrm{Irr}(R)$ is a set of normal forms for the group G, and whenever $(u, v) \in Q$ we have $u =_G v$, this shows that Q is a convergent prefix-rewriting system for the group G.

Since the convergent prefix-rewriting system R is bounded with constant k, the rules of the prefix-rewriting system Q are also bounded, with constant given by the maximum of k, 2, and $\max\{l(z_b) \mid b \in B\}$.

Note that the prefix-rewriting system Q has been chosen to satisfy properties (a), (b), and (c) of the Definition 5.1, and so Q is a processed bounded convergent prefix-rewriting system.

If moreover the set R is also synchronously regular, then the padded extension set $\mu(R) = \{\mu(u, v) \mid (u, v) \in R\}$ is a regular language over the alphabet $B_2 = (B \cup \$)^2 \setminus \{(\$, \$)\}$. Define the monoid homomorphism $\rho_1 : B_2^* \to B^*$ by $\rho((b_1, b_2)) := b_1$ if $b_1 \in B$ and $\rho((b_1, b_2)) := \lambda$ if $b_1 = \$$. The set $\mathrm{Irr}(R)$ is the language $\mathrm{Irr}(R) = A^* \setminus (\rho_1(\mu(R))A^*)$; using closure of regular languages under homomorphic image, concatenation, and complement (see Section 2.2 for more on regular languages), then $\mathrm{Irr}(R)$ is a regular set over the alphabet B. But then $\mathrm{Irr}(R)$ is also regular over any alphabet containing B, including A.

For each $b \in B$, Lemma 2.1 says that the set $L_b := \{y \mid yb \in \mathrm{Irr}(R)\}$ also is regular. Also recall from Lemma 2.2 that whenever L is a regular language over B, then the diagonal set $\Delta(L) := \{\mu(y, y) \mid y \in L\}$ is a regular language over B_2. Now the padded extension of the subset Q'' of the prefix-rewriting system Q has the decomposition

$$\mu(Q'') = \cup_{b \in B}[(\Delta(\mathrm{Irr}(R) \setminus B^*b) \cdot \mu(b^{-1}, z_b)) \cup (\Delta(L_b) \cdot \mu(bb^{-1}, \lambda))].$$

Again applying closure properties (in particular under finite unions) of regular languages, this shows that Q'' is synchronously regular.

Analyzing the subset Q' of Q requires a few more steps. First we note that the padded extension of the set of rules in R satisfying property (b) in Definition 5.1 is $\mu(R') = \mu(R) \cap \rho_1^{-1}(\mathrm{Irr}(R) \cdot B)$, and so $\mu(R')$ is a regular set.

Next for each $1 \leq i \leq n$ (where $n = |W|$), let $L_i := \rho_1(\mu(R') \cap (\Delta(B^*) \cdot \mu(s_i, t_i)))$ be the set of left hand entries of all of the rules r in R'_i. Again closure properties show that L_i is a regular language over B. Then the set

$$L'_i := L_i \setminus (\cup_{j=1}^{i-1} L_j)$$

is the set of all left hand entries of elements q in Q' such that the index $i(q) = i$. Now Lemma 2.1 shows that the set $L''_i := \{y \mid ys_i \in L'_i\}$ is regular. Putting all of these together, the padded extension of the set Q' has the decomposition

$$\mu(Q') = \cup_{i=1}^{n} \Delta(L''_i) \cdot \mu(s_i, t_i).$$

Thus $\mu(Q')$ is a regular language over the alphabet B_2, and hence also over the set A_2. Hence Q' also is synchronously regular.

Finally the closure of synchronously regular sets under finite unions shows that the bounded convergent prefix-rewriting system Q is synchronously regular, as required. □

Note that whenever R is a processed convergent prefix-rewriting system over an alphabet A and $w \in A^*$ is a reducible word, then there exists exactly one rewriting operation (of the form $w = ux \to vx$ for some $(u, v) \in R$) that can be applied to w. Hence for each $w \in A^*$, we can define the *prefix-rewriting length* $prl_R(w)$ to be the number of rewriting operations required to rewrite w to its normal form via R.

Theorem 5.3. *Let G be a finitely generated group.*
(1) The group G is stackable if and only if G admits a bounded convergent prefix-rewriting system.
(2) The group G is autostackable if and only if G admits a synchronously regular bounded convergent prefix-rewriting system.

Proof. Suppose first that the group G is stackable over an inverse-closed generating set A, with normal form set $\mathcal{N}$, constant k, and stacking function $\phi : \mathcal{N} \times A \to A^*$ such that the length of $\phi(y, a)$ is at most k for all $(y, a) \in \mathcal{N} \times A$. In Lemma 3.2, we show that

$$R_\phi := \{(ya, y\phi(y, a)) \mid y \in \mathcal{N}, a \in A, ya \notin \mathcal{N} \cup A^*a^{-1}a\}$$
$$\cup \{(yaa^{-1}, y) \mid ya \in \mathcal{N}, a \in A\}.$$

is a convergent prefix-rewriting system for the group G. (Moreover, the irreducible words are the normal forms from the stackable structure; i.e., $\mathrm{Irr}(R_\phi) = \mathcal{N}$.) The bound k on lengths of words in the image of ϕ implies that R_ϕ is a bounded convergent prefix-rewriting system.

If moreover G is autostackable, so that the set $graph(\phi)$ is synchronously regular, let $\mu(graph(\phi)) := \{\mu(y_g, a, \phi(y_g, a)) \mid g \in G, a \in A\}$ be the regular language of padded words over $A_3 = (A \cup \$)^3 \setminus \{(\$, \$, \$)\}$ associated to the elements of the set $graph(\phi)$. Define the monoid homomorphism $\rho_1 : A_3^* \to A^*$ by $\rho_1((a_1, a_2, a_3)) := a_1$ if $a_1 \in A$ and $\rho_1((a_1, a_2, a_3)) := \lambda$ if $a_1 = \$$, and the monoid homomorphism $\rho_{2,3} : A_3^* \to ((A \cup \$)^2)^*$ by $\rho_{2,3}((a_1, a_2, a_3)) := (a_2, a_3)$, for each $(a_1, a_2, a_3) \in A_3$. The normal form set $\mathrm{Irr}(R_\phi) = \mathcal{N} = \rho_1(\mu(graph(\phi)))$ is the image of a regular set, and so is regular. For each $a \in A$, the set $J_a := \{y \in A^* \mid ya \in \mathcal{N}\}$ is regular, applying Lemma 2.1. Also with this notation, for each $a \in A$ and $u \in A^{\leq k}$ we can write the set $L_{a,u}$ of all normal form words $y \in \mathcal{N}$ such that the stacking function ϕ maps (y, a) to the word u as

$$L_{a,u} := \rho_1(\mu(graph(\phi)) \cap \rho_{2,3}^{-1}(\mu(a, u) \cdot (\$, \$)^*)).$$

Recalling the fact that the class of regular languages is closed under finite intersections and homomorphic image and preimage, then since the language $\mu(a, u) \cdot (\$, \$)^*$ over $(A \cup \$)^2$ is regular, the set $L_{a,u}$ is regular. Using the notation $\Delta(L) = \{\mu(w, w) \mid w \in L\}$ for any language L, we can now decompose the padded extension of the prefix-rewriting system as

$$R_\phi = [\cup_{a \in A, u \in A^{\leq k}, a \neq u} \Delta(L_{a,u}) \cdot \mu(a, u)] \cup [\cup_{a \in A} \Delta(J_a) \cdot \mu(aa^{-1}, \$)].$$

From Lemma 2.2, the languages $\Delta(L_{a,u})$ and $\Delta(J_a)$ over $(A \cup \$)^2$ are regular. Since singleton sets are regular, and the class of regular languages is also closed under concatenation and finite unions, this decomposition shows that the set $\mu(graph(\phi))$ is regular. Therefore R_ϕ is a synchronously regular bounded convergent prefix-rewriting system for the autostackable group G.

Conversely, suppose that the group G admits a bounded convergent prefix-rewriting system. From the proof of Proposition 5.2, there exists a processed bounded convergent prefix-rewriting system R, over an inverse-closed alphabet A, for the group G. Let k be the constant associated to the bounded property of this prefix-rewriting system. Let $\mathcal{N}$ be the set $\mathrm{Irr}(R)$ of words that are irreducible with respect to the rewriting operations $ux \to vx$ whenever $(u, v) \in R$ and $x \in A^*$. Since the prefix-rewriting system is convergent, then $\mathcal{N}$ is a set of normal forms for G. Note that the empty word and any prefix of an irreducible word are irreducible, and so $\mathcal{N}$ is a prefix-closed language of normal forms for G over A that contains the empty word.

Define the function $\phi : \mathcal{N} \times A \to A^*$ as follows. For each $y \in \mathcal{N}$ and $a \in A$, define $\phi(y, a) := a$ if either $ya \in \mathcal{N}$ or $y \in A^* a^{-1}$, as required for property (2d) of Definition 1.1. If neither of these conditions hold, then the word ya is reducible. Since the maximal prefix y is irreducible, any rule of the prefix-rewriting system that applies to the word ya must have the entire word ya as its left entry. Because this prefix-rewriting system is processed, there is exactly one element of R of the form (ya, v) for some $v \in A^*$. Moreover, there are words $s, t \in A^{\leq k}$ and $w \in X^*$ such that $ya = wsa$, $v = wt$, and (by taking w to be as long as possible) the words s and t do not start with the same letter. In this case we define $\phi(y, a) := s^{-1}t$, where s^{-1} is the formal inverse of s in A^*. For every $y \in \mathcal{N}$ and $a \in A$, then, the length of the word $\phi(y, a)$ is at most $2k$, and since $wsa =_G wt$ in the rewriting presentation of G, we have $\phi(y, a) =_G a$.

Let Γ be the Cayley graph for the group G with generating set A, and let $\vec{E}_r = \vec{E}_{\mathcal{N},r}$ denote the set of recursive edges with respect to the normal form set $\mathcal{N}$. Given any directed edge $e_{g,a}$ of the Cayley graph $\Gamma(G, A)$ with $g \in G$ and $a \in A$, let $prl_R(e_{g,a}) := prl(y_g a)$ denote the prefix-rewriting length over R of the associated word $y_g a$, where y_g is the irreducible normal form for g.

Suppose that $e_{g,a}$ is any edge in $\vec{E}_r$, and that e' is an edge on the directed path in Γ labeled by the word $\phi(y_g, a)$ and starting at the vertex g. Then the word $y_g a$ is not in normal form, and there is a rule $y_g a = wsa \to wt$ in the prefix-rewriting system R such that $\phi(y_g, a) = s^{-1}t$. Since the word y_g is in normal form, the prefix s^{-1} of the word $\phi(y_g, a)$ labels a path in Γ starting at the vertex g that follows only degenerate edges, in the maximal tree defined by the normal form set $\mathcal{N}$. Writing the word $t = b_1 \cdots b_n$ with each b_i in A, then $e' = e_{gs^{-1}b_1 \cdots b_{i-1}, b_i} = e_{wb_1 \cdots b_{i-1}, b_i}$ for some i. Now the sequence of rewriting operations with respect to the prefix-rewriting system R of the word $y_g a$ has the form $y_g a = wsa \to wt = wb_1 \cdots b_n \to^* y_{gs^{-1}b_1 \cdots b_{i-1}} b_i \cdots b_n \to^* y_{ga}$, where $\to^*$ denotes a finite number (possibly 0) of applications of rewriting rules, since no rewriting operation over the processed prefix-rewriting system R can be applied affecting the letter b_i in these words until the prefix to the left of that letter has been rewritten into its irreducible normal form. Hence the number of rewritings needed to obtain an irreducible word starting from the word $y_g a$ is strictly greater than the number required to obtain a normal form starting from the word $y_{gs^{-1}b_1 \cdots b_{i-1}} b_i$. That is, $prl_R(e') < prl_R(e)$. Then the usual strict well-founded partial ordering on the natural numbers implies that the relation $<_\phi$ of property (2r) in Definition 1.1 is a strict well-founded partial ordering. Hence property (2) of the Definition 1.1 of autostackable holds, and so the group G is stackable.

If moreover G has a bounded convergent prefix-rewriting system that is synchronously regular, then Proposition 5.2 says that there is a processed synchronously regular bounded convergent prefix-rewriting system R over a inverse-closed generating set A for the group G. Synchronous regularity of R means that the set $\mu(R) = \{\mu(u, v) \mid (u, v) \in R\}$ of padded words is a regular language over the set $A_2 = (A \cup \$)^2 \setminus \{(\$, \$)\}$. Let $\rho_1 : A_2^* \to A^*$ be the monoid homomorphism defined by $\rho_1(a_1, a_2) := a_1$ if $a_1 \in A$ and $\rho_1(a_1, a_2) := \lambda$ if $a_1 = \$$. The set $\mathcal{N}$ of irreducible words with respect to R can then be written as

$$\mathcal{N} = A^* \setminus (\rho_1(\mu(R))A^*),$$

and so $\mathcal{N}$ is a regular language.

For each $a \in A$, an application of Lemma 2.1 shows that the language $L_a := \{y \mid ya \in \mathcal{N}\}$ is regular. Lemma 2.3 then shows that the languages $L_a \times \{a\} \times \{a\}$ and $(\mathcal{N} \cap A^* a^{-1}) \times \{a\} \times \{a\}$ are synchronously regular. Thus the subset of $graph(\phi)$ corresponding to the application of ϕ to degenerate edges is synchronously regular.

Given $a \in A$, let W_a be the finite set of all pairs (s,t) such that $s, t \in A^{\leq k}$, s and t begin with different letters of A, and s does not end with the letter a^{-1}. Let $\Delta(A^*) := \{(w,w) \mid w \in A^*\}$; by Lemma 2.2, this language over A_2 is regular. For each $(s,t) \in W_a$, let

$$P_{a,s,t} := \rho_1(\mu(R) \cap (\Delta(A^*) \cdot \mu(sa,t))),$$

which is again regular using the closure properties of regular languages. Then the set of all words w such that the rule (wsa, wt) lies in R is

$$L_{a,s,t} := \{w \mid wsa \in P_{a,s,t}\},$$

which is also regular (by Lemma 2.1). Applying Lemma 2.3 once more shows that the subset $(L_{a,s,t} \cdot s) \times \{a\} \times \{s^{-1}t\}$ of $graph(\phi)$ corresponding to these recursive edges is also synchronously regular.

We can now write the graph of the stacking function ϕ as

$$graph(\phi) = \cup_{a \in A}[(L_a \times \{a\} \times \{a\}) \cup ((\mathcal{N} \cap A^*a^{-1}) \times \{a\} \times \{a\})]$$
$$\cup_{a \in A, (s,t) \in W_a} (L_{a,s,t} \cdot s) \times \{a\} \times \{s^{-1}t\}.$$

Closure of the class of synchronously regular languages under finite unions then implies that $graph(\phi)$ is synchronously regular. Hence property (1) of Definition 1.1 of autostackability also holds in this case. □

Rewriting systems that are not "prefix-sensitive", allowing rewriting rules to be applied anywhere in a word, have been considerably more widely studied and applied in the literature than prefix-rewriting systems. A *finite convergent rewriting system* for a group G consists of a finite set A together with a finite subset $R \subseteq A^* \times A^*$ such that as a monoid, G is presented by $G = Mon\langle A \mid u = v \text{ whenever } u \to v \in R\rangle$, and the rewritings $xuz \to xvz$ for all $x, z \in A^*$ and (u,v) in R satisfy:

- *Normal forms:* Each $g \in G$ is represented by exactly one *irreducible* word (i.e. word that cannot be rewritten) over A.
- *Termination:* There does not exist an infinite sequence of rewritings $x \to x_1 \to x_2 \to \cdots$.

The key difference here is that a rewriting system allows rewritings $xuz \to xvz$ for all $x, z \in A^*$ and $(u,v) \in R$, but a prefix-rewriting system only allows rewritings $uz \to vz$ for all $z \in A^*$ and $(u,v) \in R$. However, every finite convergent rewriting system gives rise to a bounded convergent prefix-rewriting system, yielding the following.

Corollary 5.4. *Every group that admits a finite complete rewriting system is autostackable.*

Proof. Given a finite convergent rewriting system R for a group G over a generating set A, the prefix-rewriting system over A defined by

$$\hat{R} := \{(wu, wv) \mid (u,v) \in R, w \in A^*\}$$

allows exactly the same rewriting operations as the original finite convergent rewriting system, and therefore is a convergent prefix-rewriting system. Since the set R is finite, this prefix-rewriting system $\hat{R}$ is also bounded. Finally, the padded extension of the set $\hat{R}$ can be written as $\mu(\hat{R}) = \cup_{(u,v)\in R}\Delta(A^*)\cdot\mu(u,v)$, and so this set is synchronously regular. Theorem 5.3(2) now completes the proof. □

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