Artin groups, rewriting systems and three-manifolds

Susan M. Hermiller*

Department of Mathematical Sciences, New Mexico State University
Las Cruces, NM 88003, USA

John Meier**

Department of Mathematics, Lafayette College, Easton, PA 18042, USA

Abstract

We construct finite complete rewriting systems for two large classes of Artin groups: those of finite type, and those whose defining graphs are based on trees. The constructions in the two cases are quite different; while the construction for Artin groups of finite type uses normal forms introduced through work on complex hyperplane arrangements, the rewriting systems for Artin groups based on trees are constructed via three-manifold topology. This construction naturally leads to the question: Which Artin groups are three-manifold groups? Although we do not have a complete solution, the answer, it seems, is "not many."

1. Introduction

Let $\mathcal{G}$ be a finite simplicial graph with edges labeled by integers greater than one. Associated to $\mathcal{G}$, which we call the *defining graph*, is an infinite group $A\mathcal{G}$, whose presentation has generators corresponding to the vertices of $\mathcal{G}$, and relations

$$\underbrace{aba\cdots}_{n \text{ letters}} = \underbrace{bab\cdots}_{n \text{ letters}}$$

where $\{a, b\}$ is an edge of $\mathcal{G}$ labeled n. Such groups are *Artin groups*; typical examples are the braid groups and the fundamental groups of $(2, n)$-torus link complements. While it is relatively simple to define Artin groups, they are certainly not simple to work with. Basic questions, such as the word problem, are

* E-mail: `smh@nmsu.edu`
** E-mail: `meierj@lafayette.edu`

still open for arbitrary Artin groups, although there are certain classes of Artin groups where the word problem has been solved.

Given any Artin group $A\mathcal{G}$ there is an associated Coxeter group $C\mathcal{G}$ which is the quotient of $A\mathcal{G}$ formed by adding the relations $v^2 = 1$ for each generator v. An Artin group is of *finite type* if its associated Coxeter group is finite. Braid groups are Artin groups of finite type; their associated Coxeter groups are the symmetric groups. Thurston proved that braid groups are biautomatic (§9 in [13]), and Charney extended this result to all Artin groups of finite type [7].

Other special classes of Artin groups are known to be automatic or biautomatic. Graph groups (Artin groups where all the edge labels = 2) were shown to be biautomatic in [31] and independently in [18]. On the other end of the spectrum, Peifer has shown that Artin groups of extra-large type (all edge labels > 3) are biautomatic [25], and triangle free Artin groups ($\mathcal{G}$ does not contain a complete graph on three vertices) are known to admit automatic structures ([26] with [15]).

In this paper we present finite complete rewriting systems for two classes of Artin groups. In addition to solving the word problem, finite complete rewriting systems provide an extremely useful mechanism for converting any word in the generators into a canonical normal form.

Theorem 1. *If $A\mathcal{G}$ is an Artin group of finite type, then $A\mathcal{G}$ has a geodesic finite complete rewriting system.*

In [24] Pedersen and Yoder have independently developed (non-geodesic) finite complete rewriting systems for braid groups, using a different presentation for these groups.

Theorem 2. *If $A\mathcal{G}$ is an Artin group with defining graph $\mathcal{G}$ a tree, then $A\mathcal{G}$ has a finite complete rewriting system.*

The only other class of Artin groups which is known to admit finite complete rewriting systems are graph groups ([31] and [18]). In [18] it was shown that the class of groups admitting finite complete rewriting systems is closed under graph products. Since the graph product of Artin groups is an Artin group, there are now a large number of Artin groups which are known to admit finite complete rewriting systems. In particular, all Artin groups which can be formed by taking free and direct products of Artin groups of finite type, graph groups, and Artin groups based on trees, admit finite complete rewriting systems.

It is interesting to note that both of the classes we consider in this paper arise as fundamental groups of manifolds; those of finite type correspond to certain complex manifolds formed by taking hyperplane complements [11], and those based on trees are the fundamental groups of certain link complements [6]. This 'manifoldness' expresses itself in the proofs of Theorems 1 and 2. The proof of Theorem 1 uses normal forms introduced in [3], which are closely connected with the geometry of complex hyperplane complements, while the proof of Theorem 2 uses the fact that these Artin groups are the fundamental groups of link complements which fiber over the circle.

The connection between three-manifolds, Artin groups, and rewriting systems is intriguing because of recent work on rewriting systems and three-manifold groups. For example, in [19] it is shown that if M is a closed P^2-irreducible three-manifold with infinite fundamental group, and if $\pi_1(M)$ admits a finite complete rewriting system, then $\widetilde{M}$ is homeomorphic to $\mathbb{R}^3$; finite complete rewriting systems for many manifolds admitting one of Thurston's eight geometries are presented in [20]. Regrettably, it seems that this connection cannot be pushed much beyond the Artin groups based on trees. In particular, we suspect that the only Artin groups which are fundamental groups of compact three-manifolds are those which split as a free product, each factor being $\mathbb{Z}^3$ or having a defining graph a tree. In the final section we give good evidence for this belief by extending an argument of Droms that completely classifies which graph groups are the fundamental groups of compact three-manifolds [12], to all even Artin groups.

Theorem 3. *If $\mathcal{G}$ is an even labeled graph, then the Artin group $A\mathcal{G}$ is the fundamental group of a compact three-manifold if and only if each connected component of $\mathcal{G}$ is a tree or a triangle with all edges labeled two.*

We emphasize that by "three-manifold group" we mean "the fundamental group of a compact three-manifold"; in particular we are *not* restricting ourselves to the smaller class of closed (that is, compact without boundary) three-manifolds.

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2. Background on Rewriting Systems

A *rewriting system* for a group G consists of a finite alphabet Σ and a subset $R \subseteq \Sigma^* \times \Sigma^*$ of *rules*, where Σ^* is the free monoid on the set Σ. An element $(u, v) \in R$ is also written $u \to v$. In general, if $u \to v$, then for any $x, y \in \Sigma^*$ we write $xuy \to xvy$ and say that the word xuy is *rewritten* (or *reduced*) to the word

xvy. We also write $x \xrightarrow{*} y$ if $x \to x_1 \to x_2 \to \cdots \to y$ for some finite sequence of rewritings, including if $x = y$. The ordered pair (Σ, R) is a rewriting system for a monoid M if

$$\langle\ \Sigma \mid u = v \text{ if } (u, v) \in R\ \rangle$$

is a monoid presentation for M. A rewriting system for a group is a rewriting system for the underlying monoid. In particular, the set of elements Σ must be monoid generators for the group.

An element $x \in \Sigma^*$ is *irreducible* if it cannot be rewritten. We would like the irreducible words in Σ^* to be a set of normal forms for our group G. From a computational standpoint, we would also like to be able to start with any representative of $g \in G$ and have it be rewritten, in a finite number of steps, to the unique irreducible element representing g. These desires motivate the following definition.

Definition. A rewriting system (Σ, R) is *complete* if the following conditions hold.

C1) There is no infinite sequence $x \to x_1 \to x_2 \to \cdots$ of rewritings. (In this case the rewriting system is called *Noetherian.*)

C2) There is exactly one irreducible word representing each element of the monoid presented by the rewriting system. (Such rewriting systems are *confluent.*)

One usually establishes the Noetherian condition by imposing a well-founded ordering on Σ^*, which is compatible with concatenation, and then checking that if $u \to v \in R$, then $u > v$ in the ordering. In order to check confluence for the systems we construct, we will apply the following lemma:

Lemma 2.1. *Let Σ be a set of monoid generators for a group G, and let $\mathcal{NF} \subset \Sigma^*$ be a set of normal forms for G which is subword closed. Let R be the subset of $\Sigma^* \times \Sigma^*$ consisting of pairs of the form $u \to v$, where $u \notin \mathcal{NF}$, every proper subword of u is in $\mathcal{NF}$, $v \in \mathcal{NF}$, and $u =_G v$. Then R is a complete rewriting system if and only if it is Noetherian.*

Proof. Since every word $w \in \Sigma^*$ which is not in $\mathcal{NF}$ must contain a subword which is the left hand side of a rule in R, the irreducible words of R are exactly the normal forms in $\mathcal{NF}$. □

Finally, a rewriting system is *finite* if the set of rules R is finite, and it is *geodesic* if each irreducible word is of minimal length among all representatives of the corresponding group element. Not all groups which admit finite complete rewriting systems admit geodesic finite complete rewriting systems; if a rewriting system *is* geodesic, then the process of rewriting a word of length n is guaranteed to produce a normal form in at most $(|\Sigma|+1)^n$ steps, where Σ is the chosen set of generators [19]. (For further background, see [17] and the references cited there.) A finite complete rewriting system for a group allows the word and order problems to be solved, and it also provides an algorithm for computing the homology groups of the group. (See [10] for a survey of the connections between finite complete rewriting systems and homology.)

Given the power of finite complete rewriting systems, it is not surprising that they are difficult to construct. Some progress has been made, however, on rewriting important classes of infinite groups. Finite complete rewriting systems have been constructed for surface groups, many closed three-manifold groups admitting geometric structures, and many Coxeter groups ([17] and [20]). In this paper we will need the result that the class of groups admitting finite complete rewriting systems is closed under group extensions.

Theorem 2.2. [16] *If $1 \to K \to G \to Q \to 1$ is a short exact sequence of groups, with K and Q admitting finite complete rewriting systems, then G also admits a finite complete rewriting system.*

Since this theorem has not been published, we include a slightly revised version of the proof from [16] for completeness. We begin with a lemma.

Lemma 2.3. *Given a finite complete rewriting system (Σ, R) and a word $w \in \Sigma^*$, there is a bound on the lengths of all sequences of rewritings $w \to w_1 \to \cdots \to w_n$ (where the length of this sequence is defined to be n). There is also a bound on the length of any word to which w can be rewritten.*

Proof. Suppose that w is an element of Σ^* for which there is no such bound on the lengths of sequences of rewritings of w. Because there are only finitely many words x such that $w \to x$ with a single rewriting, there must be a word x_1 with $w \to x_1$ such that there is no bound on the length of sequences of rewritings of x_1. Repeating this argument gives a word x_2 with $x_1 \to x_2$ and no bound on the length of a sequence of rewritings of x_2, etc. This produces an infinite sequence $w \to x_1 \to x_2 \to \cdots$, contradicting the fact that R is Noetherian. Now, since there is a bound on the length of any sequence of rewritings of w, and at each stage only finitely many rewritings can be done to a single word, there are only finitely many words that appear in any of these rewritings. □

Definition. Given a word $w \in \Sigma^*$ and a finite complete rewriting system R over Σ, the *disorder* of w, denoted by $d_R(w)$, is the maximum of the lengths of all of the possible sequences of rewritings $w \to w_1 \to \cdots \to w_n$, where the *length* of this sequence is n. The *stretch* of w, denoted by $st(w)$, is the maximum of the lengths of all of the words which appear in any of these sequences. It is immediate from these definitions that $d_R(w') < d_R(w)$ and $st(w') \leq st(w)$ if $w \to w'$.

Proof of Theorem 2.2. Suppose (Σ_1, R_1) and (Σ_2, R_2) are finite complete rewriting systems for K and Q, respectively. The alphabet $\Sigma = \Sigma_1 \cup \Sigma_2$ generates G as a monoid. Define a set R_3 of rewriting rules of the form

$$kq \to q(\text{irreducible representative of } q^{-1}kq \text{ in } \Sigma_1^*),$$

for each $k \in \Sigma_1$ and $q \in \Sigma_2$, where the irreducible representative is with respect to the rewriting system R_1. Then the set $R = R_1 \cup R_2 \cup R_3$ is a finite rewriting system for G. The irreducible words, with respect to R, are exactly the words of the form vu where u is an irreducible word of the system (Σ_1, R_1) and v is an irreducible word of (Σ_2, R_2). Because there is a bijection between the elements of K and the irreducible words in (Σ_1, R_1) as well as a bijection between Q and the irreducible elements of (Σ_2, R_2), this gives a bijection between the set of irreducible words in Σ^* and G. Therefore, in order to show that R is complete, it remains to show that R is Noetherian.

If $w \in \Sigma^*$, let w' be the word in Σ_2^* obtained by deleting all letters of Σ_1 from w, and let $n = st(w')$. Then the word w can be expressed as

$$w = k_1 q_1 k_2 ... k_n q_n k_{n+1}$$

where each $k_i \in \Sigma_1^*$ and each q_i is either in Σ_2 or is empty. Also, we assume that the empty q_i are all to the right of the non-empty ones, and that any k_j to the right of an empty q_i is also empty. Define functions ψ_j from Σ^* to the non-negative integers by

$$\psi_0(w) = st(w'),$$

$$\psi_1(w) = d_{R_2}(w'),$$

$$\psi_{2i}(w) = d_{R_1}(k_i), \text{ and}$$

$$\psi_{2i+1}(w) = length(k_i),$$

where i ranges from 1 to $n+1$, and *length* denotes the word length over the alphabet Σ_1. In order to compare words of different lengths, if $j > n$, then define $\psi_j(w) = 0$.

For two words w and x in Σ^*, define $w > x$ if $\psi_0(w) > \psi_0(x)$ or if $\psi_0(w) = \psi_0(x)$ and $\psi_j(w) = \psi_j(x)$ for all $j < k$ and $\psi_k(w) > \psi_k(x)$. To see that this ordering is well-founded, notice that in an infinite chain $x_1 > x_2 > \cdots$, $\psi_j(x_i) = 0$ for all $j > 2\psi_0(x_1) + 3$ and for every index i. Then in this infinite sequence, the values of the finite set of functions $\{\psi_0, ..., \psi_{2\psi_0(x_1)+3}\}$ must all become zero after finitely many steps. However, the only word with all of the functions ψ_j equal to zero is the empty word.

Now suppose a rule in R is applied to a word $w \in \Sigma^*$. If the rule is in R_1, it must be applied to a subword k_i; this rule decreases the value of $\psi_{2i}(w)$ without altering the values of $\psi_j(w)$ for any $j < 2i$. If the rule is in R_2, then it must be applied to a subword of w' for which the intervening words k_i are empty. In this case the value of $\psi_0(w) = st(w')$ is either decreased or remains the same; however, the value of $\psi_1(w) = d_{R_2}(w)$ must decrease. Finally, if $w \to x$ by a rule in R_3, the rule is applied to a subword of w of the form $k_i q_i$, where k_i and q_i are not empty. In particular, if $k_i = \tilde{k_i} k$ and $q_i = q\tilde{q_i}$, with $\tilde{k_i} \in \Sigma_1^*$, $k \in \Sigma_1$, $q \in \Sigma_2$, and $\tilde{q_i} \in \Sigma_2^*$, then the rule replaces the subword kq by q(irreducible representative of $q^{-1}kq$ in Σ_1^*). If $w = k_1 q_1 k_2 ... k_n q_n k_{n+1}$ and $x = \hat{k}_1 \hat{q}_1 \hat{k}_2 ... \hat{k}_n \hat{q}_n \hat{k}_{n+1}$, then $\hat{q}_j = q_j$ for all j and $\hat{k}_j = k_j$ for all $j < i$. However, the word $\hat{k_i} = \tilde{k_i}$ is simply the word k_i with the last letter k removed, so $d_{R_1}(k_i) \geq d_{R_1}(\hat{k_i})$ and $length(k_i) > length(\hat{k_i})$. Thus $\psi_j(w) = \psi_j(x)$ for all $j < 2i$, $\psi_{2i}(w) \geq \psi_{2i}(x)$, and $\psi_{2i+1}(w) > \psi_{2i+1}(x)$. Thus $w > x$ and the rewriting system is Noetherian. □

3. Rewriting Artin groups of finite type

We construct two somewhat different rewriting systems for Artin groups of finite type. In both cases these rewriting systems have the added benefit that the irreducible words give relatively well understood normal forms. The first rewriting system we construct produces normal forms which are similar to those introduced in [3] and [7]; the second produces the normal forms from [8]. Charney shows in [7] and [8] that both sets of normal forms correspond to biautomatic structures for finite type Artin groups.

We should highlight the difference between Charney's biautomatic structure and these rewriting systems. A biautomatic structure gives a set of normal forms with good geometric structure, but it does not give a computationally effective procedure for converting a given word into normal form as a finite complete rewriting system does. The normal forms in both of Charney's biautomatic structures are the representatives of group elements which are minimal with respect to a shortlex ordering. In general, normal forms from a shortlex biautomatic structure would be the irreducible words of an infinite complete rewriting system; in this section we show that there are finite complete complete rewriting systems for both sets of normal forms.

In discussing Artin groups of finite type, we use extended generating sets for $A\mathcal{G}$ which are built from the elements of the associated Coxeter group. Because the word problem is relatively simple for Coxeter groups, we can easily represent the elements of $C\mathcal{G} - \{1\}$ as minimal length, or *reduced*, words in the generators corresponding to vertices in $\mathcal{G}$ (not including their inverses). If u is a minimal length word representing some nontrivial element in $C\mathcal{G}$, let $[u]$ represent the corresponding Artin group element; the word $[u]$ is exactly the same as the word u, when thought of as elements in $Vert(\mathcal{G})^*$, but as group elements they are contained in $A\mathcal{G}$ and $C\mathcal{G}$ respectively. A theorem of Tits ([5]) shows that if two reduced words u and v represent the same element of $C\mathcal{G}$, then the elements $[u]$ and $[v]$ of $A\mathcal{G}$ are also equal. We let S be the collection of all the symbols $[u]$, where the letters $[u]$ and $[v]$ are considered to be the same letter in S if u and v are reduced words representing the same element of $C\mathcal{G}$. The set S is finite, since $C\mathcal{G}$ is finite, and it generates $A\mathcal{G}$ as a group. Every finite Coxeter group contains a unique element of maximal length. This maximal length element plays a special role in both rewriting systems, so we let δ denote a representative of this 'longest element' in $C\mathcal{G}$. For every reduced word u, there are other reduced words u' and u'' so that the products $[u][u']$ and $[u''][u]$ equal $[\delta]$. (For more information on Coxeter groups, see §II.3.C of [5].)

In order to make the notation easier, the symbol [empty word] may implicitly appear on the right hand side of some rules in the rewriting systems below, or in our discussions of these systems. Since this symbol actually represents the trivial element of $A\mathcal{G}$, it should be omitted.

3.1. The first rewriting system

The alphabet in this case will be

$$\Sigma_1 = \{[u] \mid [u] \in S\} \cup \{\overline{[\delta]}\}$$

where $\overline{[\delta]}$ denotes a formal inverse for $[\delta]$; adding $\overline{[\delta]}$ gives us a set of monoid generators for $A\mathcal{G}$. In the construction of these rewriting systems we will often refer to *positive* words, which are simply non-trivial elements of the free monoid S^*.

A theorem of Deligne ([11], Prop 1.19, restated in the form we use as Lemma 2.2 in [7]), states that given any positive word α, then among all the elements

$[v] \in S$ where α is equivalent to $\gamma[v]$ in $A\mathcal{G}$ for some $\gamma \in S^*$, there is a unique maximal element $[m] \in S$ such that $\alpha = \rho[m]$. By 'maximal' we mean that whenever $\alpha = \gamma[v]$ for any $\gamma \in S^*$ and $[v] \in S$, then $[m] = [s][v]$ and $\gamma = \rho[s]$ for some word s that is either reduced or empty. Given reduced words u and v in $C\mathcal{G}$, there is a corresponding element $[m(u,v)] \in S \subset A\mathcal{G}$ that is essentially the maximal length suffix of the element $[u][v]$ of $A\mathcal{G}$ that can be represented by a reduced word of $C\mathcal{G}$. In other words, $[m(u,v)]$ is the unique element such that whenever the product $[u][v]$ equals another product $[w][x]$ in $A\mathcal{G}$, where w and x are each either a reduced or empty word, then there are also words r and s, again either reduced or empty, such that $[m(u,v)] = [s][x]$ and $[w] = [r][s]$. In particular, for a given pair of words u and v, there are words $r(u,v)$ and $s(u,v)$ such that $[m(u,v)] = [s(u,v)][v]$ and $[u] = [r(u,v)][s(u,v)]$. The elements $[m(u,v)]$, $[r(u,v)]$, and $[s(u,v)]$ depend only on the element $[u][v]$ of $A\mathcal{G}$, rather than specific choices of reduced word representatives. We use these elements to choose irreducible representatives of our rewriting system.

For each $[u] \in S$, let $[u']$ be the element for which $[\delta] = [u][u']$, and let $[\hat{u}]$ be the element for which $[\delta] = [u'][\hat{u}]$. Then u' is a reduced word representing the element $u^{-1}\delta$ in $C\mathcal{G}$, and $\hat{u}$ is a reduced word representing the element $\delta^{-1}u\delta$ in $C\mathcal{G}$.

The rules of our first rewriting system are

$$
\begin{aligned}
R_1 = \{ \quad & (1)\ [\delta]\overline{[\delta]} \to 1 && (2)\ \overline{[\delta]}[\delta] \to 1 \\
& (3)\ [u][v] \to [r(u,v)][m(u,v)] && (\text{when } s(u,v) \text{ is not empty}) \\
& (4)\ \overline{[\delta]}[u] \to [\hat{u}]\overline{[\delta]} && (\text{when } [u] \neq [\delta]) \quad \}.
\end{aligned}
$$

It is easy to check that each of the rules in R_1 is a relation in $A\mathcal{G}$ and that (Σ_1, R_1) is a rewriting system for the Artin group $A\mathcal{G}$.

This rewriting system is Noetherian, since the rules of this system are consistent with a shortlex ordering on Σ_1^*. The lexicographic ordering is established by defining $[u] < [v]$ if $\text{length}(u) < \text{length}(v)$, and $[u] < \overline{[\delta]}$ for every u. Since none of these rules allow the length of a word in Σ_1^* to increase when it is rewritten, this rewriting system is geodesic.

If the letters $[u]$ in the irreducible words of this rewriting system are replaced by shortlex minimal representatives in the vertex generating set, then the resulting words are in the set of canonical forms in [7]. While the canonical forms in [7] are not exactly normal forms, since there may be more than one canonical form corresponding to each group element, the replacement above gives a one-to-one correspondence between the irreducible words of R_1 and the subset of canonical forms in [7] which are the shortlex least for each of the group elements. Then Lemma 2.1 applies to show that R_1 is complete.

Since we choose to place maximal words $[m(u,v)]$ on the right side of the product $[u][v]$ in the third rule, the irreducible words of this rewriting system are referred to as 'right greedy'. The normal forms for Artin groups originally described in [3] were also defined in terms of the standard set of generators, not the extended set we have used here; these normal forms were 'left greedy', since they use maximal prefixes instead of maximal suffixes. Essentially the same discussion as occurs above shows that there is a 'left greedy' finite complete rewriting system on the extended generating set Σ_1 also.

Example. Let $A\mathcal{G} = \langle a, b \mid aba = bab \rangle$; this is the braid group on three strands. Our generators and rules are then given as follows.

$$\Sigma_1 = \{[a], [b], [ab], [ba], [aba] = [\delta], \overline{[aba]} = \overline{[\delta]}\}$$

$$R_1 = \{ \begin{array}{lll} [\delta]\overline{[\delta]} \to 1 & \overline{[\delta]}[\delta] \to 1 & [a][b] \to [ab] \\ [b][a] \to [ba] & [a][ba] \to [\delta] & [b][ab] \to [\delta] \\ [ab][a] \to [\delta] & [ba][b] \to [\delta] & [ab][ab] \to [a][\delta] \\ [ba][ba] \to [b][\delta] & [\delta][a] \to [b][\delta] & [\delta][b] \to [a][\delta] \\ [\delta][ab] \to [ba][\delta] & [\delta][ba] \to [ab][\delta] & \overline{[\delta]}[a] \to [b]\overline{[\delta]} \\ \overline{[\delta]}[b] \to [a]\overline{[\delta]} & \overline{[\delta]}[ab] \to [ba]\overline{[\delta]} & \overline{[\delta]}[ba] \to [ab]\overline{[\delta]} \; \} \end{array}$$

3.2. The second rewriting system

In [8], Charney describes a different regular language of normal forms for finite type Artin groups, using the alphabet

$$\Sigma_2 = \{[u], \overline{[u]} \mid [u] \in S\}.$$

In this family, the normal forms among positive words (words in the letters $[u]$ where $[u] \in S$) are the same as the ones above. However, on negative words, the normal forms are 'left greedy'. An arbitrary element of $A\mathcal{G}$ is expressed as

(normal form word in positive letters) (normal form word in negative letters).

The main advantage of this set of normal forms is that the set of generators is symmetric; that is, for each generator, there is another generator which is its inverse in the group. This is useful for relating the normal forms from the rewriting system to geometric properties, such as the growth function for the group, because the word length metric matches the metric on the Cayley graph of the group (see [8]).

To construct our second rewriting system, we will need another theorem of Deligne ([11], Prop 1.14, restated in the form we use as Lemma 2.7 in [8]). This theorem states that for any two positive words $\alpha, \beta \in S^*$, there is a unique maximal word $\nu \in S^*$ such that $\alpha = \alpha'\nu$ and $\beta = \beta'\nu$ as elements of $A\mathcal{G}$ for some $\alpha', \beta' \in S^*$; this word is 'maximal' in the sense that whenever $\alpha = \tilde{\alpha}\gamma$ and $\beta = \tilde{\beta}\gamma$ for any $\tilde{\alpha}, \tilde{\beta}, \gamma \in S^*$, then $\nu = \nu''\gamma$, $\tilde{\alpha} = \alpha''\nu''$, and $\tilde{\beta} = \beta''\nu''$ for some $\nu'', \alpha'', \beta'' \in S^*$. In particular, for each pair of reduced words u and v, there is a unique element $[n(u,v)] \in S$ such that whenever $[u] = [\tilde{u}][w]$ and $[v] = [\tilde{v}][w]$, where $\tilde{u}$, $\tilde{v}$, and w are each either a reduced or empty word, then there are also words $y(u,v)$, $z(u,v)$, and n'', again either reduced or empty, such that $[n(u,v)] = [n''][w]$, $[\tilde{u}] = [y(u,v)][n'']$, and $[\tilde{v}] = [z(u,v)][n'']$. So for these words u and v, $[u] = [y(u,v)][n(u,v)]$ and $[v] = [z(u,v)][n(u,v)]$ in $A\mathcal{G}$. The element $[n(u,v)]$ is essentially the maximal length suffix of both $[u]$ and $[v]$ in $A\mathcal{G}$ that can be represented by a reduced word of $C\mathcal{G}$. The elements $[n(u,v)]$, $[y(u,v)]$, and $[z(u,v)]$ depend only on the elements $[u]$ and $[v]$ of $A\mathcal{G}$, and not on specific choices of reduced word representatives.

For each $[u] \in S$, let $[u']$ be the element for which $[\delta] = [u][u']$.

The rules of our second rewriting system are

$$
\begin{array}{lll}
R_2 = \{ & (1)\ [u]\overline{[u]} \to 1 & (2)\ \overline{[u]}[u] \to 1 \\
& (3)\ [u][v] \to [r(u,v)][m(u,v)] & (\text{when } s(u,v) \text{ is not empty}) \\
& (4)\ \overline{[v]}\ \overline{[u]} \to \overline{[m(u,v)]}\ \overline{[r(u,v)]} & (\text{when } s(u,v) \text{ is not empty}) \\
& (5)\ [u]\overline{[v]} \to [y(u,v)]\overline{[z(u,v)]} & (\text{when } [u] \neq [v] \text{ and } n(u,v) \text{ is not empty}) \\
& (6)\ \overline{[u]}[v] \to [y(u',v')]\overline{[z(u',v')]} & (\text{when } [u] \neq [v])\ \ \}.
\end{array}
$$

To construct a well-founded ordering compatible with this rewriting system, we again use a shortlex ordering. For the lexicographic ordering, $[u] < [v]$ if $\text{length}(u) < \text{length}(v)$, $\overline{[u]} < \overline{[v]}$ if $\text{length}(u) > \text{length}(v)$, and $\overline{[u]} > [v]$ for any u and v. Since none of the rules allow word length in Σ_2^* to increase, this rewriting system is also geodesic. For this rewriting system, the irreducible words are exactly the normal forms described in [8] for these groups.

Example. We once again create a rewriting system for the braid group on three strands, $A\mathcal{G} = \langle a, b \mid aba = bab \rangle$, this time using the second rewriting system.

$$
\begin{aligned}
\mathcal{B} = \{&[a], [b], [ab], [ba], [aba] = [\delta], \\
&\overline{[a]}, \overline{[b]}, \overline{[ab]}, \overline{[ba]}, \overline{[aba]} = \overline{[\delta]}\}
\end{aligned}
$$

$$
R = \{ \quad [a]\overline{[a]} \to 1 \qquad \overline{[a]}[a] \to 1 \qquad [b]\overline{[b]} \to 1
$$

$\overline{[b]}[b] \to 1$	$[ab]\overline{[ab]} \to 1$	$\overline{[ab]}[ab] \to 1$
$[ba]\overline{[ba]} \to 1$	$\overline{[ba]}[ba] \to 1$	$[\delta]\overline{[\delta]} \to 1$
$\overline{[\delta]}[\delta] \to 1$	$[a][b] \to [ab]$	$[b][a] \to [ba]$
$[b][ab] \to [\delta]$	$[ab][a] \to [\delta]$	$[a][ba] \to [\delta]$
$[ba][b] \to [\delta]$	$\overline{[a]}\,\overline{[b]} \to \overline{[ba]}$	$\overline{[b]}\,\overline{[a]} \to \overline{[ab]}$
$\overline{[a]}\,\overline{[ab]} \to \overline{[\delta]}$	$\overline{[ab]}\,\overline{[b]} \to \overline{[\delta]}$	$\overline{[b]}\,\overline{[ba]} \to \overline{[\delta]}$
$\overline{[ba]}\,\overline{[a]} \to \overline{[\delta]}$	$[a]\overline{[ba]} \to \overline{[b]}$	$[a]\overline{[\delta]} \to \overline{[ab]}$
$[b]\overline{[ab]} \to \overline{[a]}$	$[b]\overline{[\delta]} \to \overline{[ba]}$	$[ab]\overline{[b]} \to [a]$
$[ab]\overline{[\delta]} \to \overline{[b]}$	$[ba]\overline{[a]} \to [b]$	$[ba]\overline{[\delta]} \to \overline{[a]}$
$[\delta]\overline{[a]} \to [ab]$	$[\delta]\overline{[b]} \to [ba]$	$[\delta]\overline{[ab]} \to [b]$
$[\delta]\overline{[ba]} \to [a]$	$\overline{[a]}[b] \to [ba]\overline{[ab]}$	$\overline{[a]}[ab] \to [b]$
$\overline{[a]}[ba] \to [ba]\overline{[b]}$	$\overline{[a]}[\delta] \to [ba]$	$\overline{[b]}[a] \to [ab]\overline{[ba]}$
$\overline{[b]}[ab] \to [ab]\overline{[a]}$	$\overline{[b]}[ba] \to [a]$	$\overline{[b]}[\delta] \to [ab]$
$\overline{[ab]}[a] \to \overline{[b]}$	$\overline{[ab]}[b] \to [a]\overline{[ab]}$	$\overline{[ab]}[ba] \to [a]\overline{[b]}$
$\overline{[ab]}[\delta] \to [a]$	$\overline{[ba]}[a] \to [b]\overline{[ba]}$	$\overline{[ba]}[b] \to \overline{[a]}$
$\overline{[ba]}[ab] \to [b]\overline{[a]}$	$\overline{[ba]}[\delta] \to [b]$	$\overline{[\delta]}[a] \to \overline{[ba]}$
$\overline{[\delta]}[b] \to \overline{[ab]}$	$\overline{[\delta]}[ab] \to \overline{[a]}$	$\overline{[\delta]}[ba] \to \overline{[b]}$
$[\delta][b] \to [a][\delta]$	$[\delta][a] \to [b][\delta]$	$[\delta][ba] \to [ab][\delta]$
$[\delta][ab] \to [ba][\delta]$	$[ab][ab] \to [a][\delta]$	$[ba][ba] \to [b][\delta]$
$\overline{[a]}\,\overline{[\delta]} \to \overline{[\delta]}\,\overline{[b]}$	$\overline{[b]}\,\overline{[\delta]} \to \overline{[\delta]}\,\overline{[a]}$	$\overline{[ab]}\,\overline{[\delta]} \to \overline{[\delta]}\,\overline{[ba]}$
$\overline{[ba]}\,\overline{[\delta]} \to \overline{[\delta]}\,\overline{[ab]}$	$\overline{[ab]}\,\overline{[ab]} \to \overline{[\delta]}\,\overline{[a]}$	$\overline{[ba]}\,\overline{[ba]} \to \overline{[\delta]}\,\overline{[b]}$ $\}$

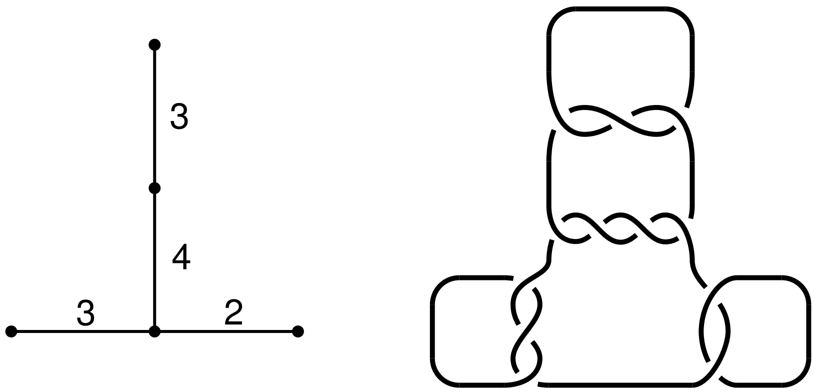
Acknowledgment. The software package Rewrite Rule Laboratory [14] aided in the analysis of the rewriting systems in these examples.

4. Rewriting Artin groups based on trees

In this and the following section, we will need a 'freiheitsatz' result of van der Lek [22]. A proof of this result, in the case where $\mathcal{G}'$ corresponds to an Artin group of finite type, has been published in [9].

Theorem 4.1. (van der Lek) *Let $\mathcal{G}$ be a labeled graph and let $\mathcal{G}'$ be a full subgraph of $\mathcal{G}$. Then the natural map $A\mathcal{G}' \to A\mathcal{G}$ is an injection.*

Every Artin group based on a tree is the fundamental group of a link exterior. To create this link, first place $\mathcal{G}$ in the plane, and put a circle around each vertex in $\mathcal{G}$. Should two vertices be joined in $\mathcal{G}$ by an edge labeled n, braid the corresponding circles together with n positive crossings. Starting from the Wirtinger presentation for this link, one can deduce that $A\mathcal{G}$ is actually the fundamental group of the resulting link exterior (see [6], especially example 2 in §6). In Figure 1 we show an example of a graph $\mathcal{G}$, and the resulting link.



A labeled graph $\mathcal{G}$ and the link corresponding to $\mathcal{G}$

FIG. 1

To see that such Artin groups admit finite complete rewriting systems, we will use the following results. The first is part of Stallings' Fibration Theorem [30]; the second follows from the partial description of the Bieri-Neumann-Strebel invariant given in [23]; and the third is a generalization of the Proposition in [12].

Theorem 4.2. (Stallings) *If G is the fundamental group of a compact three-manifold, and N is a finitely generated normal subgroup with $G/N \cong \mathbb{Z}$, then N is a surface group.*

Theorem 4.3. *Let $A\mathcal{G}$ be an Artin group based on a connected graph $\mathcal{G}$. The kernel of the map ϕ, which sends each generator of $A\mathcal{G}$ to $1 \in \mathbb{Z}$, is finitely generated.*

Lemma 4.4. (Droms) *If $G = A *_C B$, and $\phi : G \to H$ with $\phi(C) = H$, then the kernel K of ϕ can also be expressed as a free product with amalgamation: $K \cong KA *_{KC} KB$, where KA denotes $K \cap A$, etc.*

Proof. Because the argument is essentially the same as the proof in [12], we only sketch the main steps; the key idea is to use Bass-Serre theory for group actions on trees [28]. First, because G decomposes as a free product with amalgamation, G acts on a tree $\mathcal{T}$ with fundamental domain a single edge e. Since $K < G$, K also acts on $\mathcal{T}$. Further, any edge in $\mathcal{T}$ can be represented as $g \cdot e$ for some $g \in G$. Thus $(c_g g^{-1}) g \cdot e = e$ where $c_g \in C$ and $\phi(c_g) = \phi(g)$. However, $c_g g^{-1} \in K$, hence the fundamental domain for the action of K is also e. Hence K decomposes as the free product of the isotropy groups of the bounding vertices of e, amalgamating the isotropy group of e. □

Let $A\mathcal{G}$ be an Artin group based on a tree, and let $\phi : A\mathcal{G} \to \mathbb{Z}$ as in Theorem 4.3. Then the kernel K of ϕ is finitely generated, hence K is a surface group. Because surface groups admit finite complete rewriting systems, and so does $\mathbb{Z}$, by Theorem 2.1 we know that $A\mathcal{G}$ admits a finite complete rewriting system. However, rewriting systems are much more useful if they are made fairly concrete, and so we describe the structure of $A\mathcal{G}$ decomposed as a group extension in greater detail. Our proof will use the following result on one-relator groups.

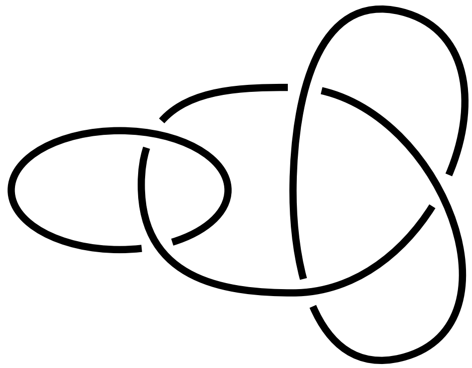
Theorem 4.5. (Bieri & Strebel, Theorem IV.5.4 in [2]) *Let G be a one-relator group $\langle x, y \mid r\rangle$ where $r = s_1 \dots s_n$ is cyclically reduced. If $\phi : G \twoheadrightarrow \mathbb{Z}$ has a finitely generated kernel, then the kernel is free of rank*

$$\max\{|\phi(s_i \dots s_j)| \mid 0 \leq i \leq j < n\} - |\phi(x)| - |\phi(y)| + 1$$

Proposition 4.6. *Let $\mathcal{G}$ be a finite, labeled simplicial tree, and let $A\mathcal{G}$ be the corresponding Artin group. Then there is a short exact sequence $1 \to F_m \to A\mathcal{G} \xrightarrow{\phi} \mathbb{Z} \to 1$ where ϕ maps each standard generator of $A\mathcal{G}$ to $1 \in \mathbb{Z}$ and the rank of the free group is $m = \sum_{e_i \in \mathcal{G}} (n_i - 1)$, where n_i is the label of the edge $e_i \in \mathcal{G}$.*

Proof. Our proof will be by induction on the size of $\mathcal{G}$. If $\mathcal{G}$ is a single edge the result follows by noting that $A\mathcal{G}$ is a one-relator group, and so by Theorem 4.5, the kernel is free of rank $n - 1$. To complete the induction, assume that the result holds for trees with fewer that m edges, and let $\mathcal{G}$ be a tree with m edges. Decompose $\mathcal{G}$ as $\mathcal{X} \cup \mathcal{Y}$, where $\mathcal{X}$ and $\mathcal{Y}$ are subtrees of $\mathcal{G}$ which intersect in a single vertex v, and each subtree has fewer than m edges. By Theorem 4.1 $A\mathcal{G}$ decomposes as a free product amalgamating the subgroup generated by v: $A\mathcal{G} \cong A\mathcal{X} *_{\mathbb{Z}} A\mathcal{Y}$. Lemma 4.4 shows that the kernel K of ϕ decomposes as a free product with amalgamation: $K \cong K\mathcal{X} *_{K\mathbb{Z}} K\mathcal{Y}$. By induction $K\mathcal{X}$ and $K\mathcal{Y}$ are free of the appropriate ranks. The result follows since ϕ restricted to $\langle v\rangle$ is an isomorphism, so $K\mathbb{Z} \cong \{id\}$. □

Example. Let $A\mathcal{G} = \langle a, b, c \mid ab = ba, bcb = cbc\rangle$. If L is the augmented trefoil knot, as in Figure 2, then $\pi_1(S^3 - L) \cong A\mathcal{G}$ [6].



The link L

FIG. 2

Let $\phi : A\mathcal{G} \to \mathbb{Z}$ be the map defined by $\phi(a) = \phi(b) = \phi(c) = 1$ as in the discussion above. Then $\langle b \rangle$ is a transversal for the kernel K of the map ϕ in $A\mathcal{G}$. The Reidemeister-Schreier process using this transversal shows that K is a free group on the generators $\{ab^{-1}, cb^{-1}, c^{-1}b\}$. If we let $x = ab^{-1}, y = cb^{-1}$ and $z = c^{-1}b$, then $\{b, \overline{b}, x, \overline{x}, y, \overline{y}, z, \overline{z}\}$ is a set of monoid generators for $A\mathcal{G}$, where letters topped with bars denote formal inverses. The general construction given by Theorem 2.1 yields the following rewriting system.

$$
\begin{array}{lllll}
R = \{ & b\overline{b} \to 1 & \overline{b}b \to 1 & x\overline{x} \to 1 & \overline{x}x \to 1 \\
 & y\overline{y} \to 1 & \overline{y}y \to 1 & z\overline{z} \to 1 & \overline{z}z \to 1 \\
 & xb \to bx & \overline{x}b \to b\overline{x} & x\overline{b} \to \overline{b}x & \overline{x}\,\overline{b} \to \overline{b}\,\overline{x} \\
 & yb \to b\overline{z} & \overline{y}b \to bz & y\overline{b} \to \overline{b}zy & \overline{y}\,\overline{b} \to \overline{b}\,\overline{y}\,\overline{z} \\
 & zb \to byz & \overline{z}b \to b\overline{z}\,\overline{y} & z\overline{b} \to \overline{b}\,\overline{y} & \overline{z}\,\overline{b} \to \overline{b}y \ \}
\end{array}
$$

5. Artin groups and three-manifolds

We remind the reader that by "three-manifold group" we mean "the fundamental group of a compact three-manifold." It was shown in [12] that a graph group is a three-manifold group if and only if each connected component of the defining graph is a tree or a triangle. In this section we show that Droms' basic argument can now be extended to a much larger class of Artin groups.

Proposition 5.1. *Let $\mathcal{G}$ be a graph with edges labeled by integers greater than one and let $A\mathcal{G}$ be the corresponding Artin group. Then*

(i) If each connected component of $\mathcal{G}$ is a tree, or a triangle with all edges labeled two, then $A\mathcal{G}$ is a three-manifold group.

(ii) If $\mathcal{G}$ is not chordal, $A\mathcal{G}$ is not coherent, hence it is not a three-manifold group.

(iii) If $\mathcal{G}$ is an even labeled graph, then $A\mathcal{G}$ is a three-manifold group if and only if each connected component of $\mathcal{G}$ is a tree or a triangle with all edges labeled two.

Recall that a graph is *chordal* if every circuit of length greater than three contains a chord, and that a group G is *coherent* if every finitely generated subgroup is finitely presented.

If $A\mathcal{G}$ is an Artin group of finite type, and δ is the special element discussed before, then δ^2 is central in $A\mathcal{G}$ (Lemma 1.26 in [11]). Since any Artin group based on a single edge is of finite type, van der Lek's theorem implies that all non-free Artin groups contain free abelian subgroups of rank two. Also, it follows from work in the previous section that, unless $A\mathcal{G}$ is free abelian, it contains non-abelian free subgroups.

Item (i) follows by taking connected sums of the manifolds corresponding to each connected component of the graph.

The proof of (ii). Suppose $\mathcal{G}$ is a non-chordal graph; then there is a full subgraph $\mathcal{C}$ of $\mathcal{G}$ which is a cycle of length greater than three. We will show that $A\mathcal{C}$ is not coherent, and hence $A\mathcal{G}$ cannot be coherent by Theorem 4.1. That three-manifold groups are coherent follows immediately from the work in [27].

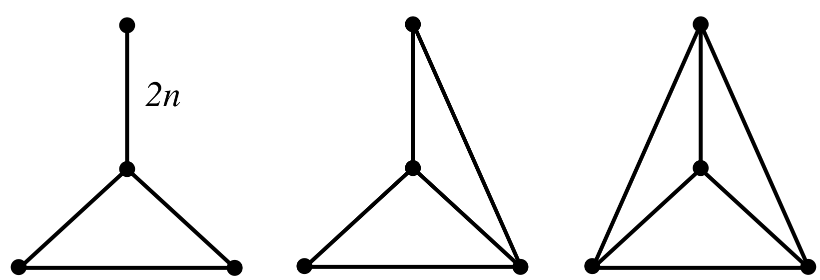
Let $\mathcal{X}$ be two adjacent edges in $\mathcal{C}$, and let $\mathcal{Y} = \overline{\mathcal{C} - \mathcal{X}}$. By Theorem 4.1, $A\mathcal{C} \cong A\mathcal{X} *_{F_2} A\mathcal{Y}$, where F_2 is the free subgroup generated by the two vertices in $\mathcal{X} \cap \mathcal{Y}$. So by Lemma 4.4, the kernel of the map $\phi : A\mathcal{C} \to \mathbb{Z}$, decomposes as a free product of $K\mathcal{X}$ and $K\mathcal{Y}$ amalgamating $K \cap F_2$. Proposition 4.6 shows that $K\mathcal{X}$ and $K\mathcal{Y}$ are finitely generated free groups. However, $K \cap F_2$ is the kernel of the induced map $F_2 \to \mathbb{Z}$, which is *not* finitely generated. The free product of finitely generated free groups, amalgamating a not-finitely generated free group, is not finitely presentable by exercise VIII.5.2 in [4], or it follows by Baumslag's more general result in [1]. Hence the kernel K is finitely generated but not finitely presented; therefore $A\mathcal{C}$ is not coherent.

The proof of (iii). We have established $\Leftarrow$ in the previous section. In order to establish $\Rightarrow$ we assume that $\mathcal{G}$ is an even-labeled graph and that $A\mathcal{G}$ is a three-manifold group. By (ii) we may assume that $\mathcal{G}$ is chordal. Our proof is by contradiction, so we also assume that some connected component of $\mathcal{G}$ is neither a tree nor a triangle with all edge labels two.

Since $\mathcal{G}$ is chordal and not a forest, $\mathcal{G}$ contains a triangle. Consider first the case in which $\mathcal{G}$ contains a triangle not all of whose edges are labeled by twos. In particular let $\Sigma \subset \mathcal{G}$ be a triangle with vertices x, y and z where the label of x—y is greater than two. By Theorem 4.1, $A\Sigma$ is a finitely presented subgroup of the compact three-manifold group $A\mathcal{G}$. A theorem of Jaco says that any finitely presented subgroup of a compact three-manifold group is itself the fundamental group of a compact three-manifold [21]. So it suffices to show that $A\Sigma$ is not the fundamental group of a compact three-manifold in order to get a contradiction in this case.

Let K be the kernel of the map $\phi : A\Sigma \to \mathbb{Z}$ defined by $\phi(x) = \phi(y) = 0$ and $\phi(z) = 1$; by work in [23], K is finitely generated. Thus, if $A\Sigma$ were a compact three-manifold group, then by Theorem 4.2, the kernel of ϕ is a surface group. However, $A\{x, y\}$ contains a copy of $\mathbb{Z}^2$ and (because the edge label of x—y is greater than two) a non-abelian free subgroup. Thus, since $A\{x, y\} < K$, the kernel K contains both free and free abelian subgroups, which is not possible if K is a surface group.

We can now assume that $\mathcal{G}$ is chordal, $\mathcal{G}$ is not a forest, and every triangle in $\mathcal{G}$ has all of its edges labeled two. It follows that one of the graphs in Figure 3 must be a subgraph of $\mathcal{G}$ [12]. (In Figure 3, all unlabeled edges are implicitly labeled "2".) Droms' work rules out the second and third possibilities. The first possibility cannot occur because the kernel of the map sending the central vertex to $1 \in \mathbb{Z}$ and all of the other vertices to 0 is finitely generated [23] and contains a copy of $\mathbb{Z}^2$ as well as a non-abelian free group. $\square$



The possible subgraphs

FIG. 3

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