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Invariants, patterns and weights for ordering terms

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We prove that any simplification order over arbtrary terms is an extension of an order by weight, by considering a related monadic term algebra called the spine. We show that any simplification order on the spine lifts to an order on the full term algebra. Conversely a simplification ordering on the term algebra defines a weight function on the spine, which in turn can be lifted to a weight order on the original ground terms which contains the original order. We investigate the Knuth Bendix and polynomial orders in this light. We provide a general framework for ordering terms by counting embedded patterns, which gives rise to many new orderings. We examine the recursive path order in this context.

1. Introduction

Orders on terms and other structures have been much studied in the context of termination proofs of rewriting systems, algorithms, logic programs and the like, and since the pioneering work of, for example, (Dershowitz, 1982), (Lescanne, 1981) and others an extraordinary variety of orders has been found: see (Steinbach, 1995) for a recent survey. More recently attention has focussed on how we might classify the orders of interest, and as a first step compute numerical or logical invariants for them. This is valuable not only for understanding more clearly the apparent diversity of these complicated structures, but also because it may help us see more readily how to try and prove termination. Furthermore in some cases these invariants give us more information about the termination problems we are trying to solve.

We call a topological space C a classifying space for a set D if there is a mapping from D to C: thus if elements of D have different images under they must be distinct. An invariant for an element of D is some number or ordinal naturally associated with elements of C, and hence under the mapping with D, so that if elements of D have distinct invariants they are distinct. Thus, for example, square matrices in one of the usual normal forms form a classifying space for linear transformations of a vector space, and the rank or the determinant are numerical invariants.

Every well-founded total order is order-isomorphic to an ordinal called its order type, which may be regarded as measuring its proof theoretic strength. If a well-order $\succ$ proves that a rewrite system $\mathcal{R}$ is terminating, then the order-type of the order can be related to the derivation complexity of $\mathcal{R}$ (Cichon and Weiermann, 1997; Hofbauer, 1992), which in turn can be related to the proof theoretic and algorithmic complexities of the relation being computed by $\mathcal{R}$ (Cichon and Weiermann, 1997). For total monotonic orders over terms in unary function symbols the order-types have been classified (Touzet, 1997;

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Hofbauer, 1992): for example, for terms over two unary function symbols, there are three possible order types (Martin and Scott, 1997).

Thus the order-type is a very weak invariant. Scott (Scott, 1994) went much further and showed that any total monotonic order on terms in unary function symbols can be linearised: that is, it is an extension of an order by weight, defined by first weighting each of the function symbols with a non-negative weight, and then weighting terms by adding up the weights of the function symbols they contain. Any pre-order by weight can be extended in continuum many ways to a total order of this kind, and different weightings give different orders provided one is not a scalar multiple of the other: in the case of two function symbols (Martin and Scott, 1997) and (Prohle and Perlo-Freeman, 1997) provide a finer classification.

It follows that any such order $\succ$ over Σ, a set of n unary function symbols, is associated with an equivalence class $p(\succ) = [(w_1, \ldots, w_n)] = \{(aw_1, \ldots, aw_n) \mid 0 < a \in \mathbb{R}\}$ of sequences of non-negative real weights. The set of all such $p(\succ)$ bijects with $\mathcal{P}_n$, the non-negative orthant of projective n-space, and this establishes $\mathcal{P}_n$ as a classifying space for these orders. This means that there is a surjective mapping from the set of orders to the classifying space, which is a topological space, and we may use this to regard the set of orders as a topological space also. Further consderation of the weights allows us to construct various invariants.

If $\mathcal{R}$ is a set of rewrite rules over Σ^* we may define a subset $\mathcal{R}_>$ of $\mathcal{P}_n$, viz the subset (possibly empty) of all those p for which any ordering $\succ$ with $p = p(\succ)$ proves $\mathcal{R}$ terminating. Thus for example for $\mathcal{R} = \{a^2 \longrightarrow b^3\}$ we have $\mathcal{R}_> = \{[(\alpha, \beta)] \mid 2\alpha > 3\beta\}$, which is an open connected subset of $\mathcal{P}_n$.

The classifying space and the associated invariants have been applied by (Martin, 1996) to classifying the rewriting systems they prove terminating: these are group and semigroup presentations. In a similar vein (Martin, 1989), (Mora and Robbiano, 1986) and others classified monotonic orders on multisets, and this classification has been used to investigate (Faugere *et al.*, 1993) the so called Gröbner walk and Gröbner fan of an ideal in a polynomial ring. This enables us to understand all Gröbner bases for an ideal in a uniform framework, and to transform one into another without recomputing from scratch. For both groups and Gröbner bases this has important practical applications, in that a rewrite system which is fast to produce (for example because it has few rules) may not be fast to compute with (for example because it has long derivation sequences) (Linton and Shand, 1996).

The results above concern numerical invariants and classifying spaces for orders on strings and multisets. This paper is concerned with developing similar results for terms. The first main section considers how we may assign numerical invariants to orders on terms, and hence establish $\mathcal{P}_n$ as a classifying space for term orders over n non-constant function symbols. The second concerns a general framework for ordering terms by counting embedded patterns: we construct a large class of new term orders and show how our method subsumes earlier constructions. A final section looks at the recursive path order in the light of our results. We now explain our results in more detail.

1.1. Linearisible orders

The first section of this paper considers how we may associate numerical invariants to term orders. That is to say, we establish a uniform framework for classifying and comparing orders on terms, with the aim of finding some analogue of Scott's results for strings. At first sight this might seem somewhat implausible: there are an enormous number of monotonic orders on terms, and it might seem unlikely we could come up with anything as elegant as Scott's result. The key is to identify a distinguished subset of the term algebra on which the orders we are considering behave like orders on strings, then to apply Scott's methods to this subset, and finally to relate the results to the original order.

As a running example throughout this section we consider the polynomial order on $T = T(\{f, g, x, y, d\})$, for f, g binary operators, d a constant and x, y variables. We define $[x] = X, [y] = Y, [d] = D, [f(x, y)] = 2XY^2$ and $[g(x, y)] = X^2 + Y^2$.

Our distinguished subset S, called the spine of T, will be defined as follows in this case, and analogously in general. Let $F(x) = f(x, x)$, $F^{i+1}(x) = f(F^i(x), F^i(x))$ and $G(x) = g(x, x)$, $G^{i+1}(x) = g(G^i(x), G^i(x))$. Then the spine of T is $S = T(\{F, G, x\})$, the set of terms on the unary operators F, G. Thus in our example the spine contains $g(f(x, x), f(x, x)) = G(F(x))$ and $f(g(x, x), g(x, x)) = F(G(x))$.

If $\succ$ is any order on T which is a total stable simplification order on restriction to S then by (Scott, 1994) there exists τ with $\infty \geq \tau \geq 0$ such that for all i, j we have

$$F^i(x) \succ G^j(x) \text{ if } i > j\tau \text{ and } F^i(x) \prec G^j(x) \text{ if } i < j\tau.$$

Further if we define $\mu : S \to \mathbb{R}$ by

$$\mu(s) = \sum_{p=1}^{k} (i_p + \tau\, j_p)$$

for

$$s = F^{i_1} G^{j_1} F^{i_2} G^{j_2} \ldots F^{i_k} G^{j_k}(x),$$

then $s \succ t$ if $\mu(s) > \mu(t)$ and $s \prec t$ if $\mu(s) < \mu(t)$. The sequence $(1, \tau)$ is a numerical invariant of the order $\succ$ on T: in general over n symbols there is a sequence of n values called the weight sequence which behaves similarly. The equivalence class of the weight sequence under scalar multiplication is an element of $\mathcal{P}_n$, establishing this as a classifying space for our class of orders.

For our running example we have

$$\begin{array}{rcl} [G(G(x))] = & [g(g(x,x), g(x,x))] & = & 8X^4 \\ & [g(g(x,x), f(x,x))] & = & 4X^6 + 4X^4 \\ [G(F(x))] = & [g(f(x,x), f(x,x))] & = & 8X^6 \\ & [f(g(x, g(x,x)), x)] & = & 8X^6 + 2X^4 \\ [F(G(x))] = & [f(g(x,x), g(x,x))] & = & 16X^6 \\ [F(F(x))] = & [f(f(x,x), f(x,x))] & = & 16X^9. \end{array}$$

Thus for $i, j \geq 1$, $F^i(x) \succ G^j(x)$ if $[F^i(x)] = aX^{3^i}$ is greater than $bX^{2^j} = [G^j(x)]$ in the polynomial ordering, where a, b are certain powers of 2. Thus $\tau = \log 2/\log 3$. The restriction of this polynomial order to S is total: it is just a certain lexicographic extension of the weight order defined by τ. The order $\succ$ corresponds to $[(1, \log 2/\log 3)]$, the class of $(1, \log 2/\log 3)$ in $\mathcal{P}_2$.

Now consider the rewrite system $\mathcal{R} = \{F(F(x)) \longrightarrow G(G(G(x)))\}$, and the subset $\mathcal{R}_>$ defined above. Our polynomial ordering lies in $\mathcal{R}_>$, and in fact $\mathcal{R}_> = \{[(1, \alpha)] \mid \alpha < 2/3\}$ a connected open subset of $\mathcal{P}_2$.

In general the description we have of the restriction of $\succ$ to S also gives us an alternative way of looking at $\succ$ on T. For it turns out that every element u of T has a least upper bound $\bar{u}$ in S, and if $\bar{u} \prec \bar{v}$ then $u \not\succ v$. Thus in our example $u = g(g(x,x), f(x,x))$ has least upper bound $\bar{u} = G(F(x))$ in S and $v = f(g(x, g(x,x)), x)$ has least upper bound $\bar{v} = F(G(x))$: the table above confirms that $u \prec \bar{u} \prec v \prec \bar{v}$.

We can go further: if we have is a total simplification order on S we can lift it under certain conditions to a simplification preorder $\trianglerighteq$ on T. If the order on S is the restriction to S of some ordering $\succ$ on T, then $\succ \subseteq \trianglerighteq$. Thus in our running example we can lift the weight-lexicographic order on S induced by the polynomial order to a preorder on T which contains the polynomial order.

In Section 3 we develop this theory precisely. The subalgebra generated by $\{F, G\}$ is an example of a so-called one parameter family in T, that is, a subalgebra that is isomorphic to a free algebra on unary function symbols. An order $\succ$ is said to be linearisible over a one parameter family S if there is a weight sequence (μ_i) satisfying the appropriate generalisation of the property of the example. For as $\succ$ is total on $S = T(\mathcal{G})$, we may assume without loss of generality that

$$\mathcal{G} = \{{}_1G, \ldots, {}_nG\}$$

where ${}_nG(x) \succ {}_{n-1}G(x) \succ \ldots \succ {}_1G(x)$. Then $\succ|_S$ is linearisible over S if there is a sequence of real numbers, called a weight sequence, satisfying

$$1 = \mu_n \geq \mu_{n-1} \geq \ldots \geq \mu_1 \geq 0$$

such that $\succ|_S \supseteq >_\mu$, where the weight function $\mu : S \to \mathbb{R}$ is defined by $\mu(s) = \sum_{{}_iG \in \mathcal{G}} \#({}_iG, s)\mu_i$ and $s >_\mu t$ if and only if $\mu(s) > \mu(t)$.

In our running example $F = {}_2G, G = {}_1G$ and $1 = \mu_2 > \mu_1 = \log 2/\log 3$ so for example $\mu(F(G(x))) = 1 + \log 2/\log 3$.

Our main theorems are

THEOREM 3.4. *Let T be a term algebra, $S = T(\mathcal{G})$ a one parameter family in T and $\succ$ a σ-stable simplification order on T which is total on restriction to S. Let $u(x), v(x)$ be elements of S. Then there exists a τ ($\infty \geq \tau \geq 0$) such that*

$$\begin{aligned} u^i(x) \succ v^j(x) \quad &\text{if} \quad i > j\tau \text{ and} \\ u^i(x) \prec v^j(x) \quad &\text{if} \quad i < j\tau \end{aligned}$$

THEOREM 3.5. *Let T be a term algebra, $S = T(\mathcal{G})$ a one parameter family in T and $\succ$ a σ-stable simplification order on T which is total on S. Then $\succ|_S$ is linearisible over S with classifying space $\mathcal{P}_n$.*

THEOREM 3.6. *Let T be a term algebra, $S = T(\mathcal{G})$ a one parameter family in T, where $\mathcal{G} = \{{}_1G(x), \ldots, {}_nG(x)\}$, and $\succ$ a σ-stable simplification order on T which is total on S. Then for each s in S there exist non-negative integers $i_1, \ldots, i_n$ and permutations π, ψ of $\{1, 2, \ldots, n\}$ such that*

$${}_{\pi(1)}G^{i_{\pi(1)}}{}_{\pi(2)}G^{i_{\pi(2)}} \ldots {}_{\pi(n)}G^{i_{\pi(n)}}(x) \succeq s \succeq {}_{\psi(1)}G^{i_{\psi(1)}}{}_{\psi(2)}G^{i_{\psi(2)}} \ldots {}_{\psi(n)}G^{i_{\psi(n)}}(x).$$

The spine, which we described informally above, is a particular example of a one parameter family, having additional properties which allow us to lift orders on the spine back to T. We have two main results of this kind:

THEOREM 3.7. *Let T be a term algebra containing constants, and let $\succ$ be a total simplification order on the spine S of T. Then there is a simplification preorder $\trianglerighteq$ on T such that if $s, t \in S$ and $s \succ t$ then $s\sigma \trianglerighteq t\sigma$ for any ground substitution σ.*

THEOREM 3.8. *Let $\succ$ be an order on T which is a total simplification order on restriction to S, preserved under ground substitutions $\sigma : \{x\} \to T$, with weight function μ on S. Suppose that there is a unique maximal constant. Then there is a function $\bar{\mu} : T \to \mathbb{R}$ such that $\succ \subseteq \geq_{\bar{\mu}}$ so that if $u \succ v$ then $\bar{\mu}(u) > \bar{\mu}(v)$.*

In our example we have $\bar{\mu}(g(g(x,x), f(x,x)) = 1 + \log 2/\log 3$.

We give further examples of one parameter families in Section 4.1, and compute weight sequences for some of the standard orders in Sections 4.2 and 4.3.

The Knuth-Bendix order is a particular order by weight defined on terms: we show that any weight sequence arises as the weight sequence of the restriction of a Knuth-Bendix order to a suitable one parameter family.

THEOREM 4.7. *Let $0 < \mu_1 \leq \ldots \leq \mu_n = 1$ and $T = T(\mathcal{F} \cup \mathcal{X})$ where $|\mathcal{F} \setminus \mathcal{F}_C| = n$ and $|\mathcal{F}_C| \geq 1$. Then there is a $\mathcal{G} \subset T$ with $|\mathcal{G}| = n$, a one parameter family $S = T(\mathcal{G} \cup \{x\})$ and a simplification order $>$ on T which is total on restriction to S and has weight sequence $(\mu_1, \ldots, \mu_n)$ over S.*

We then investigate general Knuth-Bendix orders, computing the weight sequences and, in the case that $|\mathcal{G}| = 2$, the further invariants of (Martin and Scott, 1997). The polynomial orders, first defined in (Lankford, 1975) and refined by, for example, (Ben Cherifa and Lescanne, 1987), give us a rich source of examples, and in Section 4.3 we use a technique of (Cropper, 1997) to compute the weight sequences for polynomial orders. In particular, if one of the generators of S has a non-linear interpretation then the weight sequence of S is determined solely by the degrees of the interpretations of the generators of S.

1.2. ORDERS BY COUNTING PATTERNS.

The results above concern ordering terms inside one parameter families by counting occurrences of embedded subterms of the form $F^k(x)$ for suitable patterns F. Orders by counting occurrences of various kinds of patterns, which may be subterms or some other combinatorial device, are a recurrent theme in the theory of orders, beginning with Knuth and Bendix's original paper on rewriting (Knuth and Bendix, 1970) where they described how to order terms by extending a weight function on the function symbols, and continuing with various developments by (Steinbach, 1994b), (Martin, 1993) and others.

The principle of such constructions over a term algebra T is to count occurrences of some set of patterns P_1, to order the vectors thus obtained using a division order $>_1$ on vectors, to lift this order to obtain a preorder on T, and then to break ties by considering a new set of patterns P_2. Certain compatibility conditions on the P_i are needed to ensure a simplification order is obtained. In Section 5 we present a very general framework for such constructions, which allows us to define continuum many new monotonic orders, even over a fixed signature. Essentially we count embedded subterms which are members of a pattern class, a set of variable arity terms P which is closed under taking principal subterms. The compatibility conditions are expressed by requiring that P is the disjoint union of subsets P_i, where each P_i (for $i \geq 2$) does not contain a term and any of its principal subterms, together with certain conditions on the $>_i$.

As an example consider the two terms $s = g(a, g(a, g(a, a)))$ and $t = g(g(a, a), g(a, a))$. Both contain the same number of occurrences of the varyadic subterms g, a and $g(a, a)$. However s contains more occurrences of the varyadic subterm $g(g)$ than t: our methods show that this allows us to construct a monotonic order $\succ$ with $s \succ t$, using the pattern class $\{a, g, g(g)\}$. Similarly t contains more occurrences of the varyadic subterm $g(g, g)$ than s, and we may construct a monotonic order $\succ'$ with $t \succ' s$, using the pattern class $\{a, g, g(g, g)\}$.

Section 5.1 presents the main result: proving that our construction gives rise to a monotonic order. In Section 5.2 we show the power of the new orders by showing that uncountably many distinct orders may arise in this way, even for two unary function symbols. Specifically we show that for any positive real λ there is an order $\succ_{(\lambda)}$ with $f^N g^{2M} f^N \succ_{(\lambda)} g^M f^{2N} g^M$ for $N/M > \lambda$ and $g^M f^{2N} g^M \succ_{(\lambda)} f^N g^{2M} f^N$ for $N/M < \lambda$. So for example if $\lambda = \pi$ then

$$f^{32} g^{20} f^{32} \succ_{(\lambda)} g^{10} f^{64} g^{10} \succ_{(\lambda)} f^{31} g^{20} f^{31}.$$

A suitable pattern class in this case is $\{f, g, fg, gf, fgf, gfg.\}$ The order is induced by assigning the weight $\alpha + \beta\lambda$ to a term containing α occurrences of fgf and β occurrences of gfg and comparing the weights.

We then show that the weight part of the Knuth Bendix order (Knuth and Bendix, 1970), and Martin's 'zig-zag' order (Martin, 1993) are special cases of our new construction.

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