**Math 850 Fall 2009** **Additional Practice Problems**

1. Fix $m > 0$. The $q$-ary Hamming code $C$ has parity-check matrix $H$, where $H$ is an $m \times n$ matrix whose columns are a nonzero vector in each of the 1-dimensional vector spaces in $\mathbb{F}_q^m$ (that is, no column is a multiple of another column). Determine the parameters $n, k, d$ such that $C$ is an $[n, k, d]$ code. Show that $C$ is perfect.
2. Let $C$ be an $[n, k, 3]$ GRS code with code locators $\alpha_1, \ldots, \alpha_n \in \mathbb{F}_q$ and where the column multipliers are all 1. Suppose that codeword $c$ is transmitted, $y$ is received, and the error $e = y - c$ contains one error in position $j$. Let $(S_0, S_1) = Hy^T$ be the syndrome of $y$ with respect to the canonical parity-check matrix $H$ of $C$. Show that $\alpha_j = S_1/S_0$ and that the error value $e_j$ is equal to $S_0$.
3. Show that the minimum distance of a perfect code must be odd.
4. A burst of length $\ell$ is the event of having errors in a codeword such that the locations $i$ and $j$ of the first and last errors, respectively, satisfy $j - i = \ell - 1$. Let $C$ be a linear $[n, k]$ code over $\mathbb{F}_q$ and suppose that there exists a decoder for $C$ that corrects every burst of length $t$ or less.

   (a) Show that in every nonzero codeword $c$ in $C$, the locations $i$ and $j$of the first and last nonzero entries in $c$ must satisfy $j - i \geq 2t$.

   (b) *The Reiger bound:* a Singleton-like bound for burst correcting codes. Show that $n - k \geq 2t$.

   (c) A sphere-packing-like bound for burst-correcting codes. Show that

$$q^{n-k} \geq 1 + n(q-1) + (q-1)^2 \sum_{i=0}^{t-2} (n-i-1)q^i.$$

5. Let $G$ be a $k \times n$ generator matrix of a linear code $C$ over $\mathbb{F}_q$, where $k > 0$. Show that the minimum distance of $C$ is the largest integer $d$ such that ever $k \times (n - d + 1)$ sub-matrix of $G$ has rank $k$. (*Hint*: Show that if $J$ is the set of indices of the zero entries in a nonzero codeword, then the columns in $G$ that are indexed by $J$ form a $k \times |J|$ submatrix whose rows are linearly dependent. Conversely, show that if the columns indexed by a set $J$ form a submatrix whose rows are linearly dependent, then there is a nonzero codeword in which the entries that are indexed by $J$ equal zero.)
6. Let $C$ be a linear $[n, k, d]$ code over $\mathbb{F}_q$.

   (a) Show that $C$ is MDS if and only if every set of $k$ columns in its generator matrix is linearly indepedent. (*Hint*: Use the previous problem.)

   (b) Assume that $k < n$. Show that $C$ is MDS if and only if its dual code is.