Math 850 Fall 2009 **Homework #9** Due Wed Nov 11

Warmup Problems: Section 4.2 #1,2,3,5,6,7. These are to practice concepts.
Other Problems: Section 4.2 #10,12,15,19,24. Think about some that look interesting, as time permits.
Written Problems: Do two of the following textbook problems, and both of the additional problems below. Section 4.2 #13,17,20
Additional Problems:

1. Let K be a field, and let f be an irreducible polynomial in $K[x]$. Prove that multiplicative inverses of nonzero elements exist in $K[x]/(f)$, and hence $K[x]/(f)$ is a field. (*Note*: You may assume that $K[x]$ is a Euclidean domain, and hence may use the Euclidean algorithm with respect to degree.)

2. Let K be a finite field of order q. For $n \in \mathbb{N}_0$, let a_n denote the number of monic irreducible polynomials in $K[x]$ of degree n. We will use the following algebraic fact: a monic polynomial over a field can be factored uniquely (up to the order of the factors) into monic irreducible factors.

   (a) Prove that $\langle a \rangle$ satisfies
   $$\sum_{\substack{m_1, m_2, \ldots \geq 0 \\ \sum i m_i = n}} \prod_{i \geq 1} \binom{m_i + a_i - 1}{m_i} = q^n.$$

   (b) Prove that
   $$\frac{1}{1 - qx} = \prod_{i \geq 1} (1 - t^i)^{-a_i}.$$

   (*Hint*: Interchange the order of summations in the generating function, then interchange the summation and product, and then simplify.)

   (c) Take logarithms of both sides of the identity of part (b) to obtain the relation
   $$q^n = \sum_{i \mid n} i a_i.$$

   (*Hint*: Use the Taylor series expansion of $\log(1 - x)$.) Conclude that $a_n > 0$ for all $n \geq 1$, and hence there exists an irreducible monic polynomial of any degree over any finite field. Thus, finite fields exist of order $q = p^t$ for any prime p and integer $t \geq 1$.