Math 850 Fall 2009 **Supplemental Handout #1**

Let $\mathbb{K}$ be a field.

Theorem 1. *Let f be a polynomial of degree d in $\mathbb{K}[x]$. Then f has at most d zeroes.*

Corollary 2. *If $f \in \mathbb{K}[x]$ has an infinite number of zeroes, then f is the zero polynomial.*

Theorem 3 (Combinatorial Nullstellensatz; Alon 1999)**.** *Let f be a polynomial in $\mathbb{K}[x_1,\dots,x_k]$ with degree d_i when considered as a polynomial in x_i. Let $S_1,\dots,S_k$ be sets in $\mathbb{K}$ such that $|S_i| > d_i$. If $f(z_1,\dots,z_k) = 0$ for all $(z_1,\dots,z_k) \in \prod_{i=1}^{k} S_k$, then f is the zero polynomial.*

Proof. We induct on k; the base case of $k = 1$ follows from Theorem 1. Suppose that $k > 1$, and write $f = \sum_{i=1}^{d_k} f_i x_k^i$, where $f_i \in \mathbb{K}[x_1,\dots,x_{k-1}]$. For each fixed $(k-1)$-tuple $(z_1,\dots,z_{k-1}) \in \prod_{i=1}^{k-1} S_i$, the polynomial $f(z_1,\dots,z_k) \in \mathbb{K}[x_k]$ obtained by substituting z_i for x_i ($i = 1,\dots k-1$) vanishes for all $x_k \in S_k$, and hence by Theorem 1 is the zero polynomial. Thus, the coefficient polynomial $f_i(x_1,\dots,x_{k-1}) = 0$ for all $(z_1,\dots,z_{k-1}) \in \prod_{i=1}^{k-1} S_i$. By induction, f_i is the zero polynomial for $i = 1,\dots,k-1$, and hence f is the zero polynomial. □