Math 958 Fall 2008 **Homework #3** Due Mon Dec 8

1. *Turán problem for cycles.* Fix $k \in \mathbb{N}$. Prove that there is a positive constant c_k such that for all n, there is an n-vertex graph with at least $c_k n^{1+1/(k-1)}$ edges that has no k-cycle. (*Hint*: Use the deletion method.)

2. *Coupon collector.*

   (a) Consider repeated trials of an experiment with success probability p on each trial, independently. Determine a simple formula in terms of p (not a summation) for the expected number of the trial on which the first success occurs. (*Hint*: This can be solved by computing a sum or by a shorter argument.)

   (b) Every box of a brand of candy contains one of n prizes, each with probability $1/n$. A bonus prize is won after obtaining each of these prizes at least once. Prove that the expected number of the box on which the last prize is obtained is $n\sum_{i=1}^{n} 1/i$. (*Hint*: Apply part (a).)

   (c) Every box of a brand of candy contains one of n prizes, each with probability $1/n$. Obtain a threshold for the number of boxes (as a function of n) that should be opened to obtain at least two copies of each prize. (*Hint*: In fact, adjusting a lower-order term changes the situation from almost-sure failure to almost-sure success.)

3. Let G be a digraph in which every vertex has outdegree k and indegree k. Let $r = \lfloor k/(2.25 + 2\ln k)\rfloor$. Partition $V(G)$ into r sets $V_1, \ldots, V_r$ by an appropriate experiment. Use the Local Lemma to prove that with positive probability every vertex has a successor in the set containing it. Conclude that every k-regular directed graph has a family of $\lfloor k/(2.25 + 2\ln k)\rfloor$ pairwise disjoint cycles. (*Hint*: Something must be done to ensure that the sets $V_1, \ldots, V_r$ are nonempty. *Comment*: It is conjectured that $k/2$ pairwise disjoint cycles can always be found when $\delta^+(G) \geq k$ and $k/64$ is known.)

4. *Evolution of graphs.* Let G denote a graph drawn from the standard random graph model $G_{n,p}$ generating n-vertex graphs with independent edge probability p, where p depends on n.

   (a) Prove that if $pn \to 0$, then almost always G has no cycles.

   (b) Prove that if $pn \to \infty$, then almost always G contains a cycle.