# The Ford-Fulkerson Algorithm: The Primal-Dual Algorithm Applied to Maximum Flow in a Network

Let $s$ be the source and $t$ the sink in a network. The maximum flow problem can be modeled with the following linear program D:

$$
\begin{array}{llccl}
\text{D:} & \max & \mathbf{0}'\mathbf{f} & +f_{ts} & \\
 & \text{subject to} & A\mathbf{f} & +\mathbf{e}f_{ts} & = 0 \\
 & & \mathbf{f} & & \leq \mathbf{u} \\
 & & \mathbf{f} & & \geq \mathbf{0} \\
 & & \mathbf{f} & & \text{free} \\
 & & & f_{ts} & \text{free}
\end{array}
\qquad
\begin{array}{llcccl}
\text{P:} & \min & \mathbf{0}'\mathbf{p} & +\mathbf{u}'\mathbf{q} & +\mathbf{0}'\mathbf{r} & \\
 & \text{subject to} & \mathbf{p} & & & \text{free} \\
 & & & \mathbf{q} & & \geq \mathbf{0} \\
 & & & & \mathbf{r} & \leq \mathbf{0} \\
 & & \mathbf{p}'A & +I\mathbf{q} & +I\mathbf{r} & = \mathbf{0} \\
 & & \mathbf{p}'\mathbf{e} & & & = 1
\end{array}
$$

where $A$ is the node-arc incidence matrix, $\mathbf{u}$ are the capacities on the arcs, and $\mathbf{e} = [-1, 0, 0, \ldots, 0, 1]'$ (the first row of $A$ is indexed by $s$, and the last row is indexed by $t$). The dual problem P is the minimum cut problem. Note that we are treating the constraints $\mathbf{f} \geq \mathbf{0}$ as additional constraints and not as typical nonnegativity constraints on variables. The variables $p_i$ are indexed by the nodes of the network, and the variables $q_{ij}$ and $r_{ij}$ are indexed by the arcs.

To apply the primal-dual algorithm, we think of the maximum flow problem as the dual D, and the minimum cut problem as the primal P. We start the primal-dual algorithm with a feasible flow $\mathbf{f}$. Such a flow is easy to obtain, since the zero flow $\mathbf{f} = \mathbf{0}$ is always feasible.

To determine if $\mathbf{f}$ is optimal by complementary slackness, we form the restricted primal RP. The set $J$ of admissible variables is

$$J = \{p_i : \text{for all nodes } i\} \cup \{q_{ij} : f_{ij} = u_{ij}\} \cup \{r_{ij} : f_{ij} = 0\}.$$

Hence, the restricted primal RP and its dual DRP are:

$$
\begin{array}{llccl}
\text{DRP:} & \max & \mathbf{0}'\bar{\mathbf{f}} & +\bar{f}_{ts} & \\
 & \text{s. t.} & A\bar{\mathbf{f}} & +\mathbf{e}\bar{f}_{ts} & = 0 \\
 & & \bar{f}_{ij} & & \leq 0, \text{ for } f_{ij} = u_{ij}, \\
 & & \bar{f}_{ij} & & \geq 0, \text{ for } f_{ij} = 0, \\
 & & \bar{f}_{ij} & & \leq 1 \\
 & & & \bar{f}_{ts} & \leq 1 \\
 & & \bar{\mathbf{f}} & & \text{free} \\
 & & & \bar{f}_{ts} & \text{free}
\end{array}
\qquad
\begin{array}{llccccl}
\text{RP:} & \min & & & & \mathbf{1}\mathbf{y} & +y_{ts} & \\
 & \text{s. t.} & \mathbf{p} & & & & & \text{free} \\
 & & & \mathbf{q} & & & & \geq \mathbf{0} \\
 & & & & \mathbf{r} & & & \leq \mathbf{0} \\
 & & & & & \mathbf{y} & & \geq \mathbf{0} \\
 & & & & & & y_{ts} & \geq 0 \\
 & & \mathbf{p}'A & +I\mathbf{q} & +I\mathbf{r} & +I\mathbf{y} & & = \mathbf{0} \\
 & & \mathbf{p}'\mathbf{e} & & & & +y_{ts} & = \mathbf{0} \\
 & & & q_{ij} & & & & = 0, \text{ for } f_{ij} = u_{ij}, \\
 & & & & r_{ij} & & & = 0, \text{ for } f_{ij} = 0.
\end{array}
$$

Suppose that there exists in the network (excluding the $(t, s)$ arc associated to $f_{ts}$) a path $W$ from $s$ to $t$ that satisfies

1. For arcs $(i, j)$ where $f_{ij} = u_{ij}$, the path $W$ may only traverse $(i, j)$ in the forward direction (*i.e.*, from node $i$ to node $j$).
2. For arcs $(i, j)$ where $f_{ij} = 0$, the path $W$ may only traverse $(i, j)$ in the backward direction (*i.e.*, from node $j$ to node $i$).
3. For any other arc, the path $W$ may traverse that arc either in the forward or backward direction.

Then from $W$ we can create a feasible flow $(\bar{\mathbf{f}}, \bar{f}_{ts})$ to DRP with cost 1 in the following way by setting

$$\bar{f}_{ij} = \begin{cases} 1, & \text{if } W \text{ traverses } (i, j) \text{ in the forward direction,} \\ -1, & \text{if } W \text{ traverses } (i, j) \text{ in the backward direction,} \\ 0, & \text{otherwise,} \end{cases}$$

$$f_{ts} = 1.$$

Any feasible solution $(\bar{\mathbf{f}}, \bar{f}_{ts})$ to DRP with positive cost may be used to improve the current feasible solution $(\mathbf{f}, f_{ts})$ to the maximum flow problem D. We thus set $(\hat{\mathbf{f}}, \hat{f}_{ts}) = (\mathbf{f}, f_{ts}) + \theta^*(\bar{\mathbf{f}}, \bar{f}_{ts})$, where $\theta^*$ is computed by

$$\theta^* = \min \left\{ \min_{\substack{f_{ij} < u_{ij} \\ \bar{f}_{ij} > 0}} \left[ \frac{u_{ij} - f_{ij}}{\bar{f}_{ij}} \right], \min_{\substack{f_{ij} > 0 \\ \bar{f}_{ij} < 0}} \left[ \frac{f_{ij}}{\bar{f}_{ij}} \right] \right\}.$$

Note that this computation is exactly the computation of $\theta^*$ from the primal-dual algorithm, but can easily be seen combinatorially: we wish to push $\theta^*$ additional units of flow from $s$ to $t$ along $W$. However, we cannot violate feasibility, so we choose $\theta^*$ to be the largest value so that no edge receiving increased flow violates capacity and no edge with decreased flow violates nonnegativity. Note that $\theta^* > 0$, and hence the value of the flow strictly increases.

The Ford-Fulkerson algorithm solves the maximum flow problem by iterating this procedure until no such $s, t$-path exists. At that point, we must have that the optimal cost of DRP is 0, and hence by duality the optimal cost of RP is 0. Thus, in an optimal solution to RP, $(\mathbf{y}, y_{st}) = (\mathbf{0}, 0)$. We now construct an optimal solution to RP using the fact that no such $s, t$-path exists. Let $S$ be the set of all nodes in the network reachable from the source $s$ by paths where arcs at capacity may only be used in the backward direction and arcs with zero flow may only be used in the forward direction; all other arcs may be used in either direction. Let $T$ denote the nodes not in $S$. Note that $s \in S$ and $t \in T$. Set

$$p_i = \begin{cases} 0, & \text{if } i \in S, \\ 1, & \text{if } i \in T, \end{cases} \qquad r_{ij} = \begin{cases} 1, & \text{if } i \in T \text{ and } j \in S, \\ 0, & \text{otherwise,} \end{cases}$$

$$q_{ij} = \begin{cases} 1, & \text{if } i \in S \text{ and } j \in T, \\ 0, & \text{otherwise,} \end{cases} \qquad (\mathbf{y}, y_{st}) = (\mathbf{0}, 0).$$

Then $(\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{y}, y_{st})$ is an optimal solution of RP with cost 0. Note that this solution satisfies the complementary slackness conditions for the optimal flow $f$: $q_{ij} = 0$ if $f_{ij} < u_{ij}$ and $r_{ij} = 0$ if $f_{ij} > 0$. Moreover, the optimal solution to RP is also an optimal solution to the minimum cut problem P. The cut is defined by $S$ and $T$, and the value of the cut is $\mathbf{u}'\mathbf{q}$, which is the sum of the capacities of the arcs going from $S$ to $T$. From the definition of $S$ and $T$, we have that the flow $f_{ij}$ on every arc going from $i \in S$ to $j \in T$ is at capacity and the flow $f_{ij}$ on every arc going from $i \in T$ to $j \in S$ is zero. Since the flow crossing the cut cannot return, the value of the flow $\mathbf{f}$ is $\mathbf{u}'\mathbf{q}$, which is just the statement of the Maximum Flow–Minimum Cut Theorem.

Note that since we are not actually solving DRP, we don't have a guarantee from the primal-dual algorithm that the Ford-Fulkerson algorithm will terminate. However, if all of the capacities are integral, then the algorithm will terminate in a finite number of steps since each iteration increases the value of the flow $\mathbf{f}$ by at least 1. Rational capacities may be scaled by their greatest common denominator to obtain an integral problem. However, when the capacities are not rational, the Ford-Fulkerson algorithm may not terminate in a finite number of steps and may not even converge to the maximum flow value. In this case, simple modifications to the algorithm can be made that ensure the algorithm terminates in a finite number of steps with a maximum flow.