Math 450 Combinatorics Worksheet 1

The "addition" and "multiplication" principles may be helpful in answering these questions.
Notation: $[n]$ denotes the set $\{1, 2, \ldots, n\}$.

1. How many strings of length n taken from the n letter alphabet $[n]$ are there?

2. How many strings of length k taken from the n letter alphabet $[n]$ are there?

3. How many functions $f : [s] \to [t]$ are there?

4. A function f is said to be *one-to-one* if whenever $f(x) = f(y)$, then $x = y$. (Colloquially, there are no repeated elements in the range.) How many one-to-one functions $f : [s] \to [t]$ are there? How many strings of length k taken from $[n]$ are there if no letters may be repeated?

5. An *r-permutation* is an *ordered* list of r elements taken from some set with no repetition. How many r-permutations are there taken from $[n]$? We denote this quantity by $P(n, r)$.

6. Suppose that there are 40 knights in Camelot, but King Arthur only invites 9 other knights to sit with him at the Round Table. In how many different ways can they be seated? (Note that this question is ill-posed: what makes a different seating arrangement at a round table? Two seating arrangements are different if there exists a person whose left neighbors or right neighbors are different in the two arrangements.) Such a seating arrangement is called a *circular 10-permutation*. How many circular a-permutations are there taken from $[b]$?

7. An *r-combination* is an *unordered* set of r elements taken from some set (or multiset). How many r-combinations are there taken from $[n]$? We denote this quantity by $C(n, r)$ or $\binom{n}{r}$ (the last is pronounced "n choose r").

8. A *set* is an unordered collection of objects with no repetition. In other words, we can test if an object x is in a set S or not, but there's no answer to the question "How many times does x appear in S?". A p-set is a set of size p. How many p-subsets are there of an r-set? Here, p and r are arbitrary nonzero integers; remember to consider all cases.

9. How many subsets (of any size) are there of an r-set?

10. What does the sum $\sum_{k=0}^{n} \binom{n}{k}$ equal? Why?

11. A *multiset* is an unordered collection of objects *with* repetition. In other words, to each element x there is a function $r(S, x)$ mapping into the nonzero integers that says how many times x appears in the multiset S. Suppose that $S = \{r_1 \cdot x_1, r_2 \cdot x_2, \ldots, r_k \cdot x_k\}$ is a multiset: the notation means that element x_i appears r_i times, where $r_i \geq 1$. How many permutations of S are there? (Note that a *permutation* of a set or multiset S means an n-permutation of S, where n is the number of elements in S.)

12. How many permutations are there of the letters of the word MISSISSIPPI?

13. Suppose that $S = \{r_1 \cdot x_1, r_2 \cdot x_2\}$. How many permutations of S are there? Can you put your answer in the form of $\binom{n}{r}$? Why do think this is?

14. How many 3-combinations are there of $S = \{2 \cdot a, 1 \cdot b, 3 \cdot c\}$?

15. How many r-combinations are there of a multiset S? In general, this is a hard question; we'll develop the tools to answer it later.

16. How many r-combinations are there of a multiset $S = \{\infty \cdot x_1, \infty \cdot x_2, \ldots, \infty \cdot x_k\}$ that has k *types* of objects, but infinite repetition of each type? (Consider questions 17 and 18 with this one.)

17. Consider the linear equation $y_1 + y_2 + \ldots + y_k = r$, where y_i is a nonnegative integer. How many solutions does this equation have?

18. Can you encode a specific solution to the equation in question 17 as a string of stars and bars?