Math 450 Combinatorics Homework #12 Due Wednesday, December 12

Do 5 of the following problems.

1. Prove Schur's Theorem (1916): Given $k > 0$, prove that there exists a least integer s_k such that every k-coloring of the integers $\{1, 2, \ldots, s_k\}$ yields a monochromatic x, y, z (not necessarily distinct) such that $x + y = z$. (*Hint*: Construct a graph where the vertices are the integers and the edges are appropriately colored. Then apply Ramsey's Theorem.)

2. Prove that irreducible polynomials of degree 3 exist over $\mathbb{F}_p$. (*Hint*: Things are simplified if you only consider *monic* polynomials: those polynomials whose leading coefficient is 1.)

3. 10.5.15(i,ii,iii)

4. 10.5.20 and 10.5.24

5. 10.5.28

6. A *Steiner quadruple system* (SQS) is a collection $\mathcal{Q}$ of 4-element subsets of X such that for any 3-subset T of X, there exists a unique $Q \in \mathcal{Q}$ such that $T \subseteq Q$. The integer $n = |X|$ is called the *order* of $\mathcal{Q}$.

   (a) Prove that if $\mathcal{Q}$ is an SQS of order n, then $|\mathcal{Q}| = n(n-1)(n-2)/24$.

   (b) Prove that if an SQS of order n exists, then $n \equiv 2, 4 \pmod 6$.

   (c) Let V be a (finite dimensional) vector space over $\mathbb{F}_2$ (the field with 2 elements). Define

   $$\mathcal{Q} = \left\{ Q \in \binom{V}{4} : \sum_{v \in Q} v = 0 \right\}.$$

   Prove that $\mathcal{Q}$ is an SQS on the ground set V.