Math 450 Combinatorics Homework #11 Due Wednesday, November 28

Do 5 of the following problems.

1. A *bipartite graph* is a graph $G = (V, E)$ whose vertex V is partitioned into two sets V_1 and V_2 such that edges only go between a vertex from V_1 and a vertex from V_2 (no edges are allowed between two vertices in V_1, or between two vertices in V_2). Prove, using the first moment method, that any simple graph with m edges has a bipartite subgraph containing at least $m/2$ edges.

2. 2.4.27

3. A family $\mathcal{F}$ of sets is called intersecting if $A, B \in \mathcal{F}$ implies $A \cap B \neq \emptyset$. Suppose $n \geq 2k$, and let $\mathcal{F}$ be an intersecting family of k-element subsets of the n-element set $N = \{0, 1, 2, \ldots, n-1\}$. The Erdős-Ko-Rado Theorem states that $|\mathcal{F}| \leq \binom{n-1}{k-1}$.

   (a) Given a construction of a family $\mathcal{F}$ for every n showing equality can be attained.

   (b) Prove the following: For $0 \leq s \leq n-1$, set $A_s = \{s, s+1, \ldots, s+k-1\}$ where addition is modulo n. Then $\mathcal{F}$ can contain at most k of the sets A_s. (The proof is not probabilistic, but uses the Pigeonhole Principle.)

   We prove the Erdős-Ko-Rado Theorem by calculating the probability that a randomly uniformly chosen k-subset A of N is in $\mathcal{F}$. First note that

   $$\Pr(A \in \mathcal{F}) = \frac{|\mathcal{F}|}{\binom{n}{k}}.$$

   However, we will also bound $\Pr(A \in \mathcal{F})$ by generating a randomly uniformly chosen k-subset A of N in another way. Let σ be a randomly uniformly chosen permutation of N (thought of as a bijection mapping N to itself). Let r be randomly uniformly chosen from N.

   (c) Compute $\Pr(A_r \in \mathcal{F})$.

   (d) Let $A = \{\sigma(r), \sigma(r+1), \ldots, \sigma(r+k-1)\}$. Observe that A is a k-subset randomly uniformly chosen from all k-subsets of N. Also observe that, for a fixed σ, $\Pr(A \in \mathcal{F}) = \Pr(A_r \in \mathcal{F})$. Hence, $\Pr(A \in \mathcal{F}) = \Pr(A_r \in \mathcal{F})$ (where now the probability includes σ). Use these facts to finish the proof of the Erdős-Ko-Rado Theorem.

4. Show that if the numbers $1, 2, \ldots, 9$ are colored red or blue, there will always be a 3-term arithmetic progression (3-AP) consisting of numbers of the same color. Exhibit a red/blue-coloring of the numbers $1, 2, \ldots, 8$ without any monochromatic 3-term arithmetic progression. (*Hint*: For the first part, you may assume without loss of generality that 1 is red. Now show that if 5 is also red, a 3-term monochromatic progression is forced immediately. The case when 5 is blue requires considering a few subcases.)

5. Let F be a finite collection of binary strings of finite lengths, and assume no member of F is a prefix of another one. Let N_i denote the number of strings of length i in F. Prove that

   $$\sum_{i=1}^{\infty} \frac{N_i}{2^i} \leq 1.$$

6. Suppose that in every box of your favorite cereal, there is one of n different toys. Suppose that each box contains a random toy, chosen uniformly and independently of the other boxes. You of course want every type of toy, and you keep buying boxes of cereal until you have one of every type of toy. Let B be the number of boxes that you buy. Calculate $E[B]$. (*Hint*: Think about the number of boxes you have to buy each time to get a new type of toy. The answer will be a sum of n terms.)