Math 314 Fall 2007 **Test #3** Name: Solutions

Show all of your work. Full credit may not be given for an answer alone. Clearly label the problems that you have done and your answers.

Put your name on all sheets, and put this question sheet on top of your answer sheets when you are done.

No notes, books, or papers of any other kind may be used during the exam. The last sheet of the test contains notes that you may find useful. You may remove the notes sheet from the stapled packet. The instructor has additional paper if you need it.

You may have any calculator, but **you may NOT use any function except for add, subtract, multiply, divide, exponentiate, and logarithm.** Any infraction of this rule will result in a zero for your test score.

By taking this exam you are agreeing to abide by these rules and the University of Nebraska–Lincoln Academic Integrity Policy.

Signature: ______________________________

1. (10 pts) Let $S = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 3 \\ 2 \end{bmatrix} \right\}$. Find a basis for $S^\perp$.

   Solution. We use the fact that $(\text{row}(A))^\perp = \text{null}(A)$. Thus, we put the vectors of S in as the rows of a matrix A, and a basis for the nullspace of A will be a basis for $S^\perp$.

   $$A = \begin{bmatrix} 1 & -2 & 1 & 1 \\ 1 & -1 & 3 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 5 & 3 \\ 0 & 1 & 2 & 1 \end{bmatrix}.$$

   Hence a basis for $\text{null}(A)$, and thus a basis for $S^\perp$ is

   $$\left\{ \begin{bmatrix} -5 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

   $\square$

2. (20 pts) Determine if each statement is true or false. Justify your answer if you claim the statement is true; if false, explain why the statement is false or provide an example demonstrating that it is false.

   (a) Let W be a subspace of $\mathbb{R}^n$. If $\mathbf{x}$ is not in W, then $\text{proj}_W(\mathbf{x}) = \mathbf{0}$.

   Solution. **False.** Suppose $W = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ in $\mathbb{R}^2$, and let $\mathbf{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. Clearly $\mathbf{x} \notin W$, but $\text{proj}_W(\mathbf{x}) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \neq \mathbf{0}$. $\square$

   (b) The matrix $J = \begin{bmatrix} 1 & 3 & -4 \\ 3 & 2 & 8 \\ -4 & 8 & 9 \end{bmatrix}$ is diagonalizable.

   Solution. **True.** J is a symmetric matrix with real entries, and hence by the Spectral Theorem, J is orthogonally diagonalizable. $\square$

(c) Let $A = \begin{bmatrix} 3 & 2 & -2 \\ 1 & -5 & 2 \end{bmatrix}$. The null space of A and the column space of A are orthogonal complements.

Solution. **False.** The null space of A is a subspace of $\mathbb{R}^3$, but the column space of A is a subspace of $\mathbb{R}^2$. Hence they cannot be orthogonal complements of each other. It is true that the null space of A and the row space of A are orthogonal complements in $\mathbb{R}^3$. $\square$

(d) Let $\mathcal{B} = \{1, x, x^2, x^3, \ldots\}$. Then $p(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ is in the span of $\mathcal{B}$.

Solution. **False.** The span of an *infinite* set S of vectors is the set of all linear combinations of a *finite* number of vectors from S. $p(x)$ cannot be represented as a finite linear combination of vectors in $\mathcal{B}$, and hence is not the span of $\mathcal{B}$. As an observation, note that $p(x)$ is the Taylor series of e^x, but the span of $\mathcal{B}$ is the space of all polynomials in x. $\square$

3. (20 pts) Suppose that A is a symmetric 3×3 matrix with real entries. Suppose that the eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = -2$, with corresponding eigenspaces

$$E_3 = \operatorname{span}\left\{\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}\right\} \qquad \text{and } E_{-2} = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}\right\}.$$

Orthogonally diagonalize A; that is, produce a diagonal matrix D and an orthogonal matrix Q such that $D = Q^T A Q$.

Solution. We set

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

To produce Q we need an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors of A. Eigenvectors corresponding to different eigenvectors are orthogonal. Thus, we need only produce an orthogonal basis for E_{-2}. Using the Gram-Schmidt process for

$$\left\{\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}\right\},$$

we obtain the orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2\}$, where

$$\mathbf{v}_1 = \mathbf{u}_1$$

$$\mathbf{v}_2 = \operatorname{perp}_{\operatorname{span}\{\mathbf{v}_1\}}(\mathbf{u}_2) = \mathbf{u}_2 - \operatorname{perp}_{\mathbf{v}_1}(\mathbf{u}_2) = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} - \frac{2}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

To produce an ortho*normal* basis, we must also normalize each vector to unit length. Hence,

$$Q = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{bmatrix}.$$

$\square$

4. (20 pts) Let $S = \left\{\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}\right\}.$

(a) Is S an orthogonal set? Why?

Solution. S is an orthogonal set, since

$$\begin{aligned}\mathbf{v}_1 \cdot \mathbf{v}_2 &= 2 - 4 + 2 = 0\\ \mathbf{v}_1 \cdot \mathbf{v}_3 &= -3 - 1 + 4 = 0\\ \mathbf{v}_2 \cdot \mathbf{v}_3 &= -6 + 4 + 2 = 0\end{aligned}$$

Thus, every pair of vectors in S is mutually orthogonal. □

(b) Is S an orthonormal set? Why?

Solution. S is not an orthonormal set since $\|\mathbf{v}_1\| = \sqrt{6} \neq 1$. □

(c) Is S linearly independent? Why?

Solution. S is linearly indenpedent since every orthogonal set of nonzero vectors is linearly independent. □

(d) Is S a basis for $\mathbb{R}^3$? Why?

Solution. S is linearly independent and contains 3 vectors, which is equal to $\dim \mathbb{R}^3$. Thus, S spans $\mathbb{R}^3$ and hence S is a basis. □

(e) *Without* using Gaussian elimination or the reduced row echelon form of a matrix, represent $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as a linear combination of the vectors in S.

Solution. Let $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Then

$$\begin{aligned}\mathbf{x} &= \frac{\mathbf{v}_1 \cdot \mathbf{x}}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{v}_2 \cdot \mathbf{x}}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 + \frac{\mathbf{v}_3 \cdot \mathbf{x}}{\mathbf{v}_3 \cdot \mathbf{v}_3}\mathbf{v}_3\\ &= \frac{1 - 1 + 2}{1 + 1 + 4}\mathbf{v}_1 + \frac{2 + 4 + 1}{4 + 16 + 1}\mathbf{v}_2 + \frac{-3 + 1 + 2}{9 + 1 + 4}\mathbf{v}_3\\ &= \frac{1}{3}\mathbf{v}_1 + \frac{1}{3}\mathbf{v}_2 + 0\mathbf{v}_3.\end{aligned}$$

□

5. (20 pts) Determine which of the following sets S are subspaces of the given vector space. Prove your answer.

(a) Let $\mathscr{M}_{2\times 2}$ be the space of 2×2 matrices with real entries. Let S be the subset of matrices $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ where $a_{11} + a_{22} = 0$.

Solution. S is a subspace of $\mathscr{M}_{2\times 2}$ because:

i. S is nonempty since $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S$.

ii. Let $A, B \in S$. Then $A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$, and the sum of the diagonal is

$$(a_{11} + b_{11}) + (a_{22} + b_{22}) = (a_{11} + a_{22}) + (b_{11} + b_{22}) = 0 + 0 = 0.$$

Hence, S is closed under vector addition.

iii. Let $A \in S$ and $c \in \mathbb{R}$. Then $cA = \begin{bmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{bmatrix}$, and the sum of the diagonal is

$$ca_{11} + ca_{22} = c(a_{11} + a_{22}) = c0 = 0.$$

Hence, S is also closed under scalar multiplication.

□

(b) Let $\mathbb{C}$ be the space of complex numbers, thought of as a vector space over $\mathbb{R}$. Let S be the subset of complex numbers $a + bi$ such that $ab = 1$.

Solution. S does not form a subspace, since $z = 1 + 1i \in S$, but $2z = 2 + 2i$ is not in S since $2 \cdot 2 \neq 1$. Alternatively, $z = 0 = 0 + 0i$ is not in S, and hence S can not be a subspace. □

6. (10 pts) Find a basis for $\text{span}\{\sin^2 x, \cos^2 x, \cos(2x)\}$ in $\mathscr{F}$, the space of real-valued functions on the real line. Some (possibly) helpful trigonometric identities: $\sin^2 x + \cos^2 x = 1$, $\sin(2x) = 2(\sin x)(\cos x)$, $\cos(2x) = \cos^2 x - \sin^2 x$.

Solution. Note that $\cos(2x) = \cos^2 x - \sin^2 x$, and so the set $\{\sin^2 x, \cos^2 x, \cos(2x)\}$ is linearly dependent. We remove $\cos(2x)$ and prove that $\{\sin^2 x, \cos^2 x\}$ is linearly independent. Suppose that $c_1 \sin^2 x + c_2 \cos^2 x = 0$. This equation must hold for specific values of x as well. Thus we obtain

$$\begin{aligned} x = 0 : \quad & c_1 0 + c_2 1^2 &= 0 \\ x = \tfrac{\pi}{2} : \quad & c_1 1^2 + c_2 0 &= 0 \end{aligned}$$

Thus, $c_1 = c_2 = 0$, and so $\{\sin^2 x, \cos^2 x\}$ is linearly independent. Since $\cos(2x) \in \text{span}\{\sin^2 x, \cos^2 x\}$, $\{\sin^2 x, \cos^2 x\}$ spans $\text{span}\{\sin^2 x, \cos^2 x, \cos(2x)\}$ and hence is a basis for $\text{span}\{\sin^2 x, \cos^2 x, \cos(2x)\}$.

□