Math 314 Fall 2007 **Test #2** Name: Solutions

Show all of your work. Full credit may not be given for an answer alone. Clearly label the problems that you have done and your answers.

Put your name on all sheets, and put this question sheet on top of your answer sheets when you are done.

No notes, books, or papers of any other kind may be used during the exam. The last sheet of the test contains notes that you may find useful. You may remove the notes sheet from the stapled packet. The instructor has additional paper if you need it.

You may have any calculator, but **you may NOT use any function except for add, subtract, multiply, divide, exponentiate, and logarithm.** Any infraction of this rule will result in a zero for your test score.

By taking this exam you are agreeing to abide by these rules and the University of Nebraska–Lincoln Academic Integrity Policy.

Signature: ____________________

1. (20 pts.) Let $A = \begin{bmatrix} 4 & 2 & 2 & -3 \\ 6 & -1 & 1 & 5 \\ 0 & -3 & 0 & 0 \\ 2 & -5 & 0 & 0 \end{bmatrix}$.

   (a) Compute $\det A$.

   Solution.

$$\begin{aligned}
\det A &= \det \begin{bmatrix} 4 & 2 & 2 & -3 \\ 6 & -1 & 1 & 5 \\ 0 & -3 & 0 & 0 \\ 2 & -5 & 0 & 0 \end{bmatrix} \\
&= -3 \cdot (-1)^{3+2} \det \begin{bmatrix} 4 & 2 & -3 \\ 6 & 1 & 5 \\ 2 & 0 & 0 \end{bmatrix} && \text{by expanding about the third row} \\
&= 3 \cdot 2 \cdot (-1)^{3+1} \det \begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix} && \text{by expanding about the third row} \\
&= 3 \cdot 2 \cdot (2 \cdot 5 - 1 \cdot (-3)) && \text{from our formula for det of } 2 \times 2 \text{ matrices} \\
&= 78.
\end{aligned}$$

□

   (b) Is A invertible? Why or why not?

   Solution. A is invertible because $\det A$ does not equal 0. □

2. (20 pts.) Determine if each statement is true or false. Justify your answer if you claim the statement is true; if false, explain why the statement is false or provide an example demonstrating that it is false.

   (a) If B is the 4×4 matrix obtained from $A_{4\times4}$ by adding 2 times the first row to the third row and then swapping the second and fourth rows, then $\det B = \det A$.

Solution. **False.** Performing the elementary row operation on a matrix of adding a multiple of one row to another does not change the determinant, but swapping two rows multiplies the determinant by -1. Thus, $\det B = -\det A$. If A is an invertible 4×4 matrix (for instance, $I_{4\times 4}$), then $\det A \neq 0$, and A is a counterexample to the statement. □

(b) For any $n \times n$ matrices A and B, $\det(A+B) = \det A + \det B$.

Solution. **False.** Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.Then $\det A = \det B = 0$, but $\det(A+B) = \det I_{2\times 2} = 1$. □

(c) If 0 is an eigenvalue of A, then $\det A = 0$.

Solution. **True.** If λ is an eigenvalue of A, then $\det(A - \lambda I) = 0$. Thus,

$$\det(A - 0I) = \det(A) = 0.$$

□

(d) If A and B are $n \times n$ matrices with the same reduced row echelon form, then A and B are similar.

Solution. **False.** Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Both A and B are invertible, and so have the same reduced row echelon form, namely the identity I. However, A and B are not similar, since 2 is an eigenvalue of B but it is not an eigenvalue of A. □

3. (25 pts.) Determine if each matrix is diagonalizable. If the matrix is diagonalizable, find the matrices P and D such that the matrix is equal to PDP^{-1}. If the matrix is not diagonalizable, explain why not.

(a) $A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$

Solution. The characteristic polynomial of A is

$$\begin{aligned} \det(A - \lambda I) &= \det\begin{bmatrix} 1-\lambda & 1 \\ -1 & 3-\lambda \end{bmatrix} \\ &= (1-\lambda)(3-\lambda) - 1 \\ &= \lambda^2 - 4\lambda + 4 \\ &= (\lambda - 2)^2. \end{aligned}$$

Thus, 2 is an eigenvalue of multiplicity 2. We now wish to determine if the dimension of the eigenspace corresponding to 2 has the same dimension as the multiplicity. The dimension of the eigenspace is the dimension of the nullpsace$(A - 2I)$, which is just nullity $(A - 2I)$. Since we have

$$A - 2I = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix},$$

nullity $(A - 2I) = 1$. But the multiplicity of 2 is 2. Thus, we will not be able to find a basis for $\mathbb{R}^2$ consisting of eigenvectors of A, and so A is not diagonalizable. □

(b) $B = \begin{bmatrix} 7 & 5 \\ -10 & -8 \end{bmatrix}$

Solution. The characteristic polynomial of B is

$$\begin{aligned}\det(B-\lambda I) &= \det\begin{bmatrix} 7-\lambda & 5 \\ -10 & -8-\lambda \end{bmatrix} \\ &= (7-\lambda)(-8-\lambda)-(-50) \\ &= \lambda^2+\lambda-6 \\ &= (\lambda+3)(\lambda-2).\end{aligned}$$

Thus, the eigenvalues of B are 2 and -3, both of multiplicity 1. Since B has n (in this case $n = 2$) distinct eigenvalues, B is diagonalizable. To form the matrix P, we need to find eigenvectors that correspond to each of the eigenvalues. For the eigenvalue 2, such an eigenvector is a solution to the system $(B-2I)\mathbf{x} = \mathbf{0}$.

$$B-2I = \begin{bmatrix} 5 & 5 \\ -10 & -10 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Thus, a basis for the eigenspace is

$$\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}.$$

For the eigenvalue -3, we want to find a solution to the system $(B+3I)\mathbf{x} = \mathbf{0}$.

$$B+3I = \begin{bmatrix} 10 & 5 \\ -10 & -5 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix}$$

Thus, a basis for the eigenspace is

$$\left\{\begin{bmatrix} -1/2 \\ 1 \end{bmatrix}\right\}.$$

We form the matrix P by putting the eigenvectors as the columns of P, and we form the diagonal matrix D by putting the corresponding eigenvalues down the diagonal. Thus,

$$P = \begin{bmatrix} -1 & -1/2 \\ 1 & 1 \end{bmatrix}, \text{ and } \quad D = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}.$$

□

(*there are problems on the back*)

4. (20 pts.) Let $A = \begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix}$ be the transition matrix of a Markov chain. Find the stationary distribution for the Markov chain.

Solution. We know that 1 is an eigenvalue of A. To find an eigenvector corresponding to 1, we wish to solve $(A-I)\mathbf{x} = \mathbf{0}$.

$$A-I = \begin{bmatrix} -0.4 & 0.1 \\ 0.4 & -0.1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -.25 \\ 0 & 0 \end{bmatrix}$$

Thus, $\left\{\begin{bmatrix} .25 \\ 1 \end{bmatrix}\right\}$ is a basis for the eigenspace corresponding to the eigenvalue 1. A probability vector has the property that the sum of its components is 1. To find a probability vector in the eigenspace, we scale any eigenvector by 1 over the sum of its components:

$$\mathbf{p} = \frac{1}{.25+1}\begin{bmatrix} .25 \\ 1 \end{bmatrix} = \begin{bmatrix} .2 \\ .8 \end{bmatrix}.$$

□

5. (15 pts.) If B is invertible, prove that $\det(B^{-1}AB) = \det A$.

Solution.

$$\begin{aligned}
\det(B^{-1}AB) &= \det(B^{-1})\det(A)\det(B) && \text{since det is multiplicative} \\
&= \frac{1}{\det(B)}\det(A)\det(B) \\
&= \frac{1}{\det(B)}\det(B)\det(A) && \text{scalars can be rearranged} \\
&= \det A.
\end{aligned}$$

□