Math 314 Fall 2007 **Test #1** Name: <u>Solutions</u>

Show all of your work. Full credit may not be given for an answer alone. Clearly label the problems you have done and your answers.

Put your name on all sheets, and put this question sheet on top of your answer sheets when you are done.

No notes, books, or papers of any other kind may be used during the exam.

You may have any calculator, but **you may NOT use any function except for add, subtract, multiply, divide, exponentiate, and logarithm.** Any infraction of this rule will result in a zero for your test score.

By taking this exam you are agreeing to abide by these rules and the University of Nebraska–Lincoln Academic Integrity Policy.

Signature: ____

1. (10 pts.) Solve the following system of equations. Write the general solution in vector form.

$$\begin{array}{rrrrcr} x_1 & -x_2 & +x_3 & & = & -4 \\ x_1 & -x_2 & +2x_3 & +2x_4 & = & -5 \\ 3x_1 & -3x_2 & +2x_3 & -2x_4 & = & -11 \end{array}$$

Solution. We proceed by forming the augmented matrix $[A\ \mathbf{b}]$ for the system $A\mathbf{x} = \mathbf{b}$ given above.

$$[A\ \mathbf{b}] = \begin{bmatrix} 1 & -1 & 1 & 0 & -4 \\ 1 & -1 & 2 & 2 & -5 \\ 3 & -3 & 2 & -2 & -11 \end{bmatrix}$$

We then put $[A\ \mathbf{b}]$ into reduced row echelon form.

$$\begin{bmatrix} 1 & -1 & 1 & 0 & -4 \\ 1 & -1 & 2 & 2 & -5 \\ 3 & -3 & 2 & -2 & -11 \end{bmatrix} \xrightarrow[r_3-3r_1]{r_2-r_1} \begin{bmatrix} 1 & -1 & 1 & 0 & -4 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & -1 & -2 & 1 \end{bmatrix} \xrightarrow[r_3+r_2]{r_1-r_2} \begin{bmatrix} 1 & -1 & 0 & -2 & -3 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Writing the corresponding linear system of equations, we have

$$\begin{array}{rcl} x_1 - x_2 - 2x_4 & = & -3 \\ x_2 & & \text{free} \\ x_3 + 2x_4 & = & -1 \\ x_4 & & \text{free} \end{array}$$

Here x_1 and x_3 are the basic variables, and x_2 and x_4 are free variables. Writing the general solution in vector form,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3 + x_2 + 2x_4 \\ x_2 \\ -1 - 2x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

□

2. (16 pts.) Let R be the reduced row echelon form of A.

$$A = \begin{bmatrix} 1 & 2 & -3 & 4 & 3 \\ -2 & -4 & 9 & -7 & -3 \\ 1 & 2 & 0 & 5 & 5 \end{bmatrix}, \qquad R = \begin{bmatrix} 1 & 2 & 0 & 5 & 0 \\ 0 & 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) Find bases for the row space, column space, and null space of A.

Solution. A basis for the row space of A is given by the nonzero rows of R:

$$\left\{ \begin{bmatrix} 1 & 2 & 0 & 5 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 1/3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right\}$$

A basis for the column space of A is given by the columns of A that correspond to the columns of R that have leading 1s:

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 9 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 5 \end{bmatrix} \right\}$$

A basis for the null space of A is given by writing the solution of $A\mathbf{x} = \mathbf{0}$ in vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 5x_4 \\ x_2 \\ -\frac{1}{3}x_4 \\ x_4 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -5 \\ 0 \\ -\frac{1}{3} \\ 1 \\ 0 \end{bmatrix} x_4$$

Hence a basis for the null space is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ -\frac{1}{3} \\ 1 \\ 0 \end{bmatrix} \right\}$. □

(b) Is the system $A\mathbf{x} = \mathbf{b}$ consistent, where $\mathbf{b} = \begin{bmatrix} 3 \\ 4 \\ -7 \end{bmatrix}$? If the system is consistent, does it have more than one solution?

Solution. $A_{m\times n}\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{R}^m$ if and only if rank $A = m$. Since the rank of the A given above is 3, $A_{m\times n}\mathbf{x} = \begin{bmatrix} 3 \\ 4 \\ -7 \end{bmatrix}$ is consistent. Since the system is consistent, the nullity of A is the number of free variables. The nullity of A is positive, so there are an infinite number of solutions. □

3. (24 pts.) Give a short (1 or 2 sentences) answer to each question.

(a) What are the possible number of solutions of a linear system of equations over the reals?

Solution. There are either 0, 1, or an infinite number of solutions. □

(b) Define linear independence.

Solution. A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ in $\mathbb{R}^n$ is called linearly independent if the only solution to the equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_k\mathbf{v}_k = \mathbf{0}$ is the trivial solution $c_1 = c_2 = \dots = c_k = 0$. □

(c) Is $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(\mathbf{x}) = \mathbf{x}^T\mathbf{x}$ a linear transformation? Justify your answer!

Solution. Note that $\mathbf{x}^T$ denotes the transpose of $\mathbf{x}$, where $\mathbf{x}$ is a column vector. T is not a linear transformation: Let $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. Then $T(\mathbf{x}+\mathbf{y}) = 9 \neq 5 = T(\mathbf{x}) + T(\mathbf{y})$. □

(d) True or false? (Justify your answer!) Let A be an $m \times n$ matrix. If $\mathbf{w}$ is in the column space of A, then $A\mathbf{w} = \mathbf{0}$.

Solution. False. If $m \neq n$, then $A\mathbf{w}$ doesn't even make sense. The solutions of $A\mathbf{x} = \mathbf{0}$ form the null space, not the column space. □

(e) True or false? (Justify your answer!) Suppose that $T : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear transformation represented by the matrix A (*i.e.*, $T(\mathbf{x}) = A\mathbf{x}$). Then the columns of A are linearly independent.

Solution. True. Since T is invertible, A is an invertible matrix, and hence its columns are linearly independent. □

(f) True or False? (Justify your answer!) A linear system with fewer equations than variables always has solutions.

Solution. False. Consider the system

$$\begin{aligned} x_1 \quad &= 1 \\ x_1 \quad &= 2 \\ x_1 + x_2 + x_3 + x_4 &= 5 \end{aligned}$$

Though this system has 3 equations and 4 variables, it has no solution. □

4. (15 pts.) Let $A = \begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & -1 \\ 2 & 0 & -3 \end{bmatrix}$.

(a) Find A^{-1}.

Solution. We put the matrix $\begin{bmatrix} A & I \end{bmatrix}$ into reduced row echelon form.

$$\left[\begin{array}{rrrrrr} -1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & -1 & 0 & 1 & 0 \\ 2 & 0 & -3 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrrrrr} 1 & 0 & 0 & 6 & -3 & 5 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 4 & -2 & 3 \end{array}\right]$$

Hence, $A^{-1} = \begin{bmatrix} 6 & -3 & 5 \\ -1 & 1 & -1 \\ 4 & -2 & 3 \end{bmatrix}$. □

(b) Use part (a) to solve $A\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Solution. We compute $\mathbf{x} = A^{-1}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ -1 \\ 5 \end{bmatrix}$. □

5. (15 pts.) Let $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ be two bases of $\mathbb{R}^2$.

(a) Find the coordinate vector of $\mathbf{x} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ with respect to the basis $\mathcal{B}$.

Solution. Let B be the matrix whose columns are the vectors of $\mathcal{B}$. Then we solve $B[\mathbf{x}]_{\mathcal{B}} = \mathbf{x}$:

$$\begin{bmatrix} 2 & 0 & 4 \\ 1 & 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

Thus, $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. ☐

(b) Find the change-of-basis matrix $\mathcal{P}_{\mathcal{C}\leftarrow\mathcal{B}}$ for changing from coordinates with respect to the basis $\mathcal{B}$ to coordinates with respect to the basis $\mathcal{C}$.

Solution. Let B be as defined above, and let C be the matrix whose columns are the vectors of $\mathcal{C}$. Then we form $\begin{bmatrix} C & B \end{bmatrix}$ and find the reduced row echelon form.

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} I & \mathcal{P}_{\mathcal{C}\leftarrow\mathcal{B}} \end{bmatrix}$$

Hence, $\mathcal{P}_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. ☐

(c) Using $\mathcal{P}_{\mathcal{C}\leftarrow\mathcal{B}}$, find the coordinate vector of $\mathbf{x}$ with respect to the basis $\mathcal{C}$.

Solution. We compute

$$[\mathbf{x}]_{\mathcal{C}} = \mathcal{P}_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

☐

6. Prove the following statements.

(a) (8 pts.) Suppose that A and B are $n \times n$ matrices and that $\text{rank}(A) = \text{rank}(B) = n$. Show that $\text{rank}(AB) = n$.

Solution. Since $\text{rank}(A) = \text{rank}(B) = n$, both A and B are invertible matrices. Hence, AB is invertible (with inverse $B^{-1}A^{-1}$) and has rank n. ☐

(b) (12 pts.) Suppose that $\mathcal{B} = \{\mathbf{u}, \mathbf{v}\}$ is a basis for a subspace S of $\mathbb{R}^n$. Prove that $\{\mathbf{u}, \mathbf{u} + \mathbf{v}\}$ is also a basis for S.

Solution. Let $\mathcal{C} = \{\mathbf{u}, \mathbf{u} + \mathbf{v}\}$. We wish to show that $\text{span}(\mathcal{C}) = S$. Let $\mathbf{w} \in S$. Then $\mathbf{w} = c_1\mathbf{u} + c_2\mathbf{v} = (c_1 - c_2)\mathbf{u} + c_2(\mathbf{u} + \mathbf{v})$. So $\mathbf{w} \in \text{span}(\mathcal{C})$. Hence $\mathcal{C}$ spans S.

We also wish to show that $\mathcal{C}$ is linearly independent. Suppose that $d_1\mathbf{u} + d_2(\mathbf{u} + \mathbf{v}) = \mathbf{0}$. But then $(d_1 + d_2)\mathbf{u} + d_2\mathbf{v} = \mathbf{0}$, and since $\mathcal{B} = \{\mathbf{u}, \mathbf{v}\}$ is a basis and thus linearly independent, we have that $d_1 + d_2 = 0$ and $d_2 = 0$. But this implies that $d_1 = d_2 = 0$. So $\mathcal{C}$ is linearly independent. ☐