Math 314 Fall 2007 **Quiz #9** Name: Solutions

Show all of your work; answers without work may receive no credit. Make sure that you answer each question completely.

Let

$$\mathbf{u}_1 = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}.$$

1. (3 pts) Verify that $S = \{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal set.

   Solution. We compute

   $$\mathbf{u}_1 \cdot \mathbf{u}_2 = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2}\left(-\frac{1}{2}\right) = 0.$$

   Thus, S is an orthogonal set. □

2. (4 pts) Is S an orthonormal set? If not, scale each vector in S by its length to produce an orthonormal basis S'.

   Solution. We determine the length of each vector in S:

   $$\|\mathbf{u}_1\| = \sqrt{\mathbf{u}_1 \cdot \mathbf{u}_1} = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}.$$

   $$\|\mathbf{u}_2\| = \sqrt{\mathbf{u}_2 \cdot \mathbf{u}_2} = \sqrt{\left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}.$$

   Thus, S is not an orthonormal set, since not all of the vectors in S have length 1. We form S' from S as follows:

   $$S' = \left\{\frac{1}{\|\mathbf{u}_1\|}\mathbf{u}_1, \frac{1}{\|\mathbf{u}_2\|}\mathbf{u}_2\right\} = \left\{\sqrt{2}\mathbf{u}_1, \sqrt{2}\mathbf{u}_2\right\} = \left\{\begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}, \begin{bmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}\right\}.$$

   □

3. (3 pts) Write $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as a linear combination of vectors in S'. Recall that if W is a subspace with orthonormal basis $\{\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_k\}$, then for $\mathbf{w} \in W$,

   $$\mathbf{w} = \left(\frac{\mathbf{q}_1 \cdot \mathbf{w}}{\mathbf{q}_1 \cdot \mathbf{q}_1}\right)\mathbf{q}_1 + \left(\frac{\mathbf{q}_2 \cdot \mathbf{w}}{\mathbf{q}_2 \cdot \mathbf{q}_2}\right)\mathbf{q}_2 + \ldots + \left(\frac{\mathbf{q}_k \cdot \mathbf{w}}{\mathbf{q}_k \cdot \mathbf{q}_k}\right)\mathbf{q}_k.$$

   Solution. Let $S' = \{\mathbf{q}_1, \mathbf{q}_2\}$. We compute

   $$\begin{aligned} \mathbf{v} &= \left(\frac{\mathbf{q}_1 \cdot \mathbf{v}}{\mathbf{q}_1 \cdot \mathbf{q}_1}\right)\mathbf{q}_1 + \left(\frac{\mathbf{q}_2 \cdot \mathbf{v}}{\mathbf{q}_2 \cdot \mathbf{q}_2}\right)\mathbf{q}_2 \\ &= \left(\frac{\sqrt{2}/2}{1}\right)\begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} + \left(\frac{\sqrt{2}/2}{1}\right)\begin{bmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}. \end{aligned}$$

   □