Math 314 Fall 2007 **Quiz #11** Name: Solutions

Show all of your work; answers without work may receive no credit. Make sure that you answer each question completely.

Suppose that A and B are orthogonally diagonalizable $n \times n$ matrices with real entries.

1. (5 pts) Use the Spectral Theorem to prove that $A + B$ is orthogonally diagonalizable.

   Solution. By the Spectral Theorem, an $n \times n$ matrix with real entries is orthogonally diagonalizable if and only if the matrix is symmetric. Hence, both A and B is symmetric. Since the symmetric $n \times n$ matrices form a vector space (and hence has closure of addition), $A + B$ is also a symmetric matrix, and thus is also orthogonally diagonalizable. $\square$

2. (5 pts) Suppose that A is invertible. Prove that A^{-1} is orthogonally diagonalizable. (*Hint*: Use the definition of orthogonally diagonalizable.)

   Solution. Since A is orthogonally diagonalizable, there exists an $n \times n$ orthogonal matrix Q and a diagonal matrix D such that $D = Q^T AQ$. Taking the inverse of both sides, we obtain

$$D^{-1} = \left(Q^T AQ\right)^{-1} = Q^{-1}A^{-1}\left(Q^T\right)^{-1}.$$

   But since Q is orthogonal, $Q^{-1} = Q^T$, and so $D^{-1} = Q^T A^{-1} Q$. Note that if

$$D = \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{bmatrix}, \qquad \text{then } D^{-1} = \begin{bmatrix} \frac{1}{d_{11}} & 0 & \cdots & 0 \\ 0 & \frac{1}{d_{22}} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{d_{nn}} \end{bmatrix}.$$

   Hence, D^{-1} is a diagonal matrix, and so A is orthogonally diagonalizable. $\square$