Math 314 Fall 2007 **Project** Due Wednesday, December 12, 2007

A linear system of differential equations can be written as

$$\frac{d\vec{x}(t)}{dt} = A\vec{x}(t), \qquad \vec{x}(0) = \vec{x}_0, \tag{1}$$

where A is an n by n matrix and $\vec{x}(t), \vec{x}_0 \in \mathbb{R}^n$. It is well known that if a is a scalar, then the solution to the scalar equation

$$\frac{dx(t)}{dt} = ax(t), \qquad x(0) = x_0$$

is $x(t) = e^{at}x_0$. Therefore we would hope that the solution to (1) is $\vec{x}(t) = e^{At}\vec{x}_0$, provided that e^{At} is defined properly. Let

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k}{k!} t^k.$$

This series converges for all A and t; you don't need to prove this.

1. Using the series definition of e^{At}, compute e^{At} for the following matrices:

$$A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ -1 & 3 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

   You can use a calculator to help obtain A^n, or to help identify the pattern for A^n (although it's not necessary).

2. In the book (p. 343–345) it is shown how to obtain e^{At} when A is diagonalizable. Find e^{At} for

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 0 & 3 \\ 0 & -3 & 0 \end{bmatrix}.$$

3. Prove that for any matrix A,

$$\frac{d}{dt}e^{At} = Ae^{At}.$$

   You can differentiate the series term-by-term without worrying about convergence.

4. We need some definitions here. Let $\mathbf{0}$ denote the zero matrix, and we say that a matrix function $M(t)$ satisfies

$$\lim_{t\to\infty} M(t) = \mathbf{0}$$

   if each entry $m_{i,j}(t)$ satisfies

$$\lim_{t\to\infty} m_{i,j}(t) = 0.$$

   We say that e^{At} is *stable* if

$$\lim_{t\to\infty} e^{At} = \mathbf{0}.$$

   Note that if e^{At} is stable, then all components of the solution $\vec{x}(t)$ of (1) decay to 0 as time goes on.

   (a) Prove that if A is diagonalizable and all eigenvalues $\lambda = a + ib$ of A have $a < 0$, then e^{At} is stable. *Hint*: Recall the equality

$$e^{(c+id)} = e^c(\cos d + i \sin d).$$

(b) Prove that if A has an eigenvalue $\lambda = a + ib$ with $a \geq 0$, then e^{At} is not stable. *Hint*: Look at $e^{At}\vec{v}$, where $\vec{v}$ is the eigenvector associated with λ.

5. Consider the system of differential equations

$$\frac{dx}{dt} = y,$$

$$\frac{dy}{dt} = \alpha x - 2y.$$

Find all real parameters α (excluding $\alpha = -1$) for which $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} y(t) = 0$. Why can't we use the results from Question 4. when $\alpha = -1$?

Guidelines

Groups: You can work on this project in groups of size 1, 2, or 3. If you are working in a group of size at least 2, please email me with the group members by Friday, November 16.

Computers: You can conceivably do this project using just a fancy calculator. I encourage you, however, to use the software package Maple (or an equivalent one). For one thing, it is far easier to enter, manipulate, and view large amounts of data in Maple than it is on a calculator, and this project will require the use of rather long calculations. Also, I believe it is a valuable experience to learn how to use a computer algebra system such as Maple. Maple is available on the Mathlab computer system and everyone in class should have an account on Mathlab. The Mathlab is located in room 18 of Avery Hall and you can see what its hours are at the website:

`http://www.math.unl.edu/resources/computer/labs.shtml`

Your login and password for the computers in the Mathlab should be the same as for your usual university "active directory" account.

Important Writing Instructions: After you have analyzed the mathematics of solving systems of linear differential equations and the matrix exponential, write a well-written report summarizing your findings. This report should be written to form a clear, well-organized description of what you have learned, accessible to someone who knows nothing about the specfiic project you have been assigned. In particular, do not treat the above itemized list as some sort of worksheet to be filled in with unconnected answers. Instead, strive to write a report that stands on its own and can be read and understood by someone who is not taking our class. To be more specific, *pretend the reader of your report is someone who has taken Math 314 in the distant past and who does not have a copy of this assignment*. In particular, *do not write this as if I am the only one reading it!*

Grading: This project is worth 100 points. Of these, 75 points will be for mathematical content. The remaining 25 points will be based on clarity of exposition. I will take off for things like poor grammar, spelling errors, awkward style, unclear writing, etc. I encourage you to type your report, but neatly handwritten reports will also be accepted.

Deadline: The final deadline for this project is Wednesday, December 12. This is about four weeks away, but I would strongly encourage you not to put it off. In particular, if you give me a rough draft well before the due date, I will look it over and offer feedback.