Math 314 Fall 2007 **Practice Test #2** Name: Solutions

Show all of your work. Full credit may not be given for an answer alone. Clearly label the problems that you have done and your answers.

Put your name on all sheets, and put this question sheet on top of your answer sheets when you are done.

No notes, books, or papers of any other kind may be used during the exam. The last sheet of the test contains notes that you may find useful. You may remove the notes sheet from the stapled packet. The instructor has additional paper if you need it.

You may have any calculator, but **you may NOT use any function except for add, subtract, multiply, divide, exponentiate, and logarithm.** Any infraction of this rule will result in a zero for your test score.

By taking this exam you are agreeing to abide by these rules and the University of Nebraska–Lincoln Academic Integrity Policy.

Signature: ________________________________

1. (15 pts.) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 1 \\ 1 & 6 & 4 \end{bmatrix}$.

   (a) Compute $\det A$.

   Solution.

$$\begin{aligned} \det A &= \det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 1 \\ 1 & 6 & 4 \end{bmatrix} \\ &= (-1)^{1+1} 1 \det \begin{bmatrix} 4 & 1 \\ 6 & 4 \end{bmatrix} + (-1)^{(1+3)} 1 \det \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} && \text{by expanding about the first column} \\ &= (16-6) + (2-12) && \text{from our formula for det of } 2 \times 2 \text{ matrices} \\ &= 0. \end{aligned}$$

   ☐

   (b) Is A invertible? Why or why not?

   Solution. A is not invertible since $\det A = 0$. ☐

2. (20 pts.) Determine if each statement is true or false. Justify your answer if you claim the statement is true; if false, explain why the statement is false or provide an example demonstrating that it is false.

   (a) If λ is an eigenvalue of A, then λ^2 is an eigenvalue of A^2.

   Solution. **True.** Since λ is an eigenvalue of A, there is a nonzero eigenvector $\mathbf{v}$ such that $A\mathbf{v} = \lambda \mathbf{v}$. Then

$$A^2\mathbf{v} = A(A\mathbf{v}) = A(\lambda \mathbf{v}) = \lambda A \mathbf{v} = \lambda^2 \mathbf{v}.$$

   Thus, λ^2 is an eigenvalue of A^2. ☐

   (b) For any $n \times n$ matrices A and B, $\det(A+B) = \det A + \det B$.

   Solution. **False.** Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then $\det A = \det B = 0$, but $\det(A+B) = \det I_{2\times 2} = 1$. ☐

(c) If V and W are subspaces of $\mathbb{R}^n$ having the same dimension, then $V = W$.

Solution. **False.** Consider $V = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$ and $W = \operatorname{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$. Both V and W are subspaces of $\mathbb{R}^2$ with dimension 1, yet $V \neq W$ since $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is in V but not in W. □

(d) If A and B are $n \times n$ matrices with the same n distinct eigenvalues, then A is similar to B.

Solution. **True.** Since both A and B have n distinct eigenvalues, both A and B are diagonalizable. Then $P^{-1}AP = D = Q^{-1}BQ$, where D is the diagonal matrix with the same eigenvalues along the diagonal, and P and Q are $n \times n$ invertible matrices. Hence, $B = QP^{-1}APQ^{-1} = (PQ^{-1})^{-1}A(PQ^{-1})$, and so B is similar to A. □

3. (15 pts) Use the adjoint to compute the inverse of the following matrix.

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

Solution. $A^{-1} = \frac{1}{\det A}\operatorname{adj}(A)$, where the i, j entry of $\operatorname{adj}(A)$ is $(-1)^{i+j}\det(A_{ji})$. Hence,

$$A^{-1} = \frac{1}{(0+1+1)-(-1-1+0)} \begin{bmatrix} 1 & (-1)1 & 2 \\ (-1)(-1) & 1 & (-1)2 \\ -2 & (-1)(-2) & 0 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & -2 \\ -2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}.$$

□

4. (15 pts) Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$, and suppose that $\det A = 2$.

(a) Compute $\det \begin{bmatrix} a+g & b+h & c+i \\ g & h & i \\ d & e & f \end{bmatrix}$.

Solution. This matrix is formed from A by adding a multiple of one row to another and then switching two rows, so the determinant is -2. □

(b) Suppose that B is a 3×3 matrix with determinant -2. Compute $\det(B^{-1}A)$.

Solution. $\det(B^{-1}A) = \det(B^{-1})\det(A) = \frac{1}{\det(B)}\det(A) = \frac{1}{-2}(2) = -1$. □

5. (20 pts) Let $A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. Find the eigenvalues of A, and find bases for the corresponding eigenspaces.

Solution. The characteristic polynomial is

$$\begin{aligned}\det(A-\lambda I) &= \det\begin{bmatrix} 1-\lambda & 2 & 0 \\ -1 & -1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{bmatrix} \\ &= (1-\lambda)^2(-1-\lambda) - (1-\lambda) + 2(1-\lambda) \\ &= (1-\lambda)\left[(-1+\lambda^2) - 1 + 2\right] \\ &= (1-\lambda)\lambda^2.\end{aligned}$$

Hence, the eigenvalues are 0 and 1.

To find bases for the eigenspace, we solve $(A-\lambda I)\mathbf{x} = \mathbf{0}$. For $\lambda = 0$, we have

$$A - 0I = A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{so} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} x_3.$$

Hence,

$$E_0 = \operatorname{span}\left\{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right\}.$$

For $\lambda = 1$, we have

$$A - 1I = \begin{bmatrix} 0 & 2 & 0 \\ -1 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{so} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x_3.$$

Hence, $E_1 = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$. □

6. (15 pts) If B is invertible, prove that $\det(B^{-1}AB) = \det A$.

   Solution.

$$\begin{aligned}\det(B^{-1}AB) &= \det(B^{-1})\det(A)\det(B) \\ &= \frac{1}{\det(B)}\det(A)\det(B) \\ &= \frac{1}{\det(B)}\det(B)\det(A) \\ &= \det A.\end{aligned}$$

□