Math 314 Fall 2007 **Practice Test #1** Name: Solutions

Show all of your work. Full credit may not be given for an answer alone.

No notes, books, or papers of any other kind may be used during the exam.

You may have any calculator, but **you may NOT use any function except for add, subtract, multiply, divide, exponentiate, and logarithm.** Any infraction of this rule will result in a zero for your test score.

By taking this exam you are agreeing to abide by these rules and the University of Nebraska–Lincoln Academic Integrity Policy.

Signature: ______________________________

Note: This is a test from an old semester, and was designed for one hour. Our test will be designed for 50 minutes. This test also does not cover change of basis, coordinates, or linear transformations, which are topics covered by our test.

1. (16 pts.) Let $A = \begin{bmatrix} 1 & 2 & -3 & 4 & 3 \\ -2 & -4 & 9 & -7 & -3 \\ 1 & 2 & 0 & 5 & 5 \end{bmatrix}$.

   (a) Find the reduced row echelon form of A.

   Solution.

$$A = \begin{bmatrix} 1 & 2 & -3 & 4 & 3 \\ -2 & -4 & 9 & -7 & -3 \\ 1 & 2 & 0 & 5 & 5 \end{bmatrix} \xrightarrow{\text{swap } R_1 \text{ and } R_2} \begin{bmatrix} 1 & 2 & 0 & 5 & 5 \\ -2 & -4 & 9 & -7 & -3 \\ 1 & 2 & -3 & 4 & 3 \end{bmatrix}$$

$$\xrightarrow[R_3 - R_1]{R_2 + 2R_1} \begin{bmatrix} 1 & 2 & 0 & 5 & 5 \\ 0 & 0 & 9 & 3 & 7 \\ 0 & 0 & -3 & -1 & -2 \end{bmatrix} \xrightarrow{\frac{1}{9}R_2} \begin{bmatrix} 1 & 2 & 0 & 5 & 5 \\ 0 & 0 & 1 & 1/3 & 7/9 \\ 0 & 0 & -3 & -1 & -2 \end{bmatrix}$$

$$\xrightarrow{R_3 + 3R_2} \begin{bmatrix} 1 & 2 & 0 & 5 & 5 \\ 0 & 0 & 1 & 1/3 & 7/9 \\ 0 & 0 & 0 & 0 & 1/3 \end{bmatrix} \xrightarrow{3R_3} \begin{bmatrix} 1 & 2 & 0 & 5 & 5 \\ 0 & 0 & 1 & 1/3 & 7/9 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[R_2 - 7/9R_3]{R_1 - 5R_3} \begin{bmatrix} 1 & 2 & 0 & 5 & 0 \\ 0 & 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \text{rref of } A$$

□

   (b) What are the rank of A and the nullity of A?

   Solution. The rank of A is the number of nonzero rows in the reduced row echelon form of A. Thus, rank $A = 3$. Since rank(A) + nullity(A) is the number of columns of A, the nullity of A is 2. □

   (c) Is the system $A\mathbf{x} = \mathbf{b}$ consistent, where $\mathbf{b} = \begin{bmatrix} 3 \\ 4 \\ -7 \end{bmatrix}$? If the system is consistent, does it have more than one solution?

   Solution. $A_{m\times n}\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{R}^m$ if and only if rank $A = m$. Since the rank of the A given above is 3, $A_{m\times n}\mathbf{x} = \begin{bmatrix} 3 \\ 4 \\ -7 \end{bmatrix}$ is consistent. Since the system is consistent, the nullity of A is the number of free variables. The nullity of A is positive, so there are an infinite number of solutions. □

2. (24 pts.) Determine if each statement is true or false. Justify your answer if you claim the statement is true; if false, explain why the statement is false or provide an example demonstrating that it is false.

(a) If an $n \times n$ matrix A is invertible, then $A\mathbf{x} = \mathbf{b}$ is consistent for any $\mathbf{b}$ in $\mathbb{R}^n$.

Solution. **True.** Multiplying the equation $A\mathbf{x} = \mathbf{b}$ on the left on both sides by A^{-1}, we have that $\mathbf{x} = A^{-1}\mathbf{b}$, which is clearly defined for any $\mathbf{b}$ in $\mathbb{R}^n$. Thus, $A\mathbf{x} = \mathbf{b}$ is consistent for any $\mathbf{b}$ in $\mathbb{R}^n$. □

(b) A subset of $\mathbb{R}^n$ containing fewer than n vectors must be linearly independent

Solution. **False.** The set $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right\}$ is a subset of $\mathbb{R}^3$ with only 2 vectors, but is linearly dependent since $2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = \mathbf{0}$. □

(c) There exists a 5×8 matrix with rank 3 and nullity 2.

Solution. **False.** The sum of the rank and the nullity is equal to the number of columns of the matrix, which in this case is 8. □

(d) The columns of an invertible matrix are linearly independent.

Solution. **True.** The reduced row echelon form of an invertible matrix A is the identity matrix I. The columns of I are clearly linearly independent, and by the Linear Correspondence Property, so are the columns of A. □

(e) If S_1 and S_2 are finite subsets of $\mathbb{R}^n$ having equal spans, then $S_1 = S_2$.

Solution. **False.** $S_1 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ and $S_2 = \left\{ \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right\}$ are finite subsets of $\mathbb{R}^3$ with equal spans, but $S_1 \neq S_2$. □

(f) A linear system with fewer equations than variables always has solutions.

Solution. **False.** Consider the system

$$\begin{aligned} x_1 \phantom{{}+x_2+x_3+x_4} &= 1 \\ x_1 \phantom{{}+x_2+x_3+x_4} &= 2 \\ x_1 + x_2 + x_3 + x_4 &= 5 \end{aligned}$$

Though this system has 3 equations and 4 variables, it has no solution. □

3. (10 pts.) Solve the following system of equations. Write the general solution in vector form.

$$\begin{aligned} x_1 - x_2 + x_3 \phantom{{}+2x_4} &= -4 \\ x_1 - x_2 + 2x_3 + 2x_4 &= -5 \\ 3x_1 - 3x_2 + 2x_3 - 2x_4 &= -11 \end{aligned}$$

Solution. We proceed by forming the augmented matrix $[A\ \mathbf{b}]$ for the system $A\mathbf{x} = \mathbf{b}$ given above.

$$[A\ \mathbf{b}] = \begin{bmatrix} 1 & -1 & 1 & 0 & -4 \\ 1 & -1 & 2 & 2 & -5 \\ 3 & -3 & 2 & -2 & -11 \end{bmatrix}$$

We then put $[A\ \mathbf{b}]$ into reduced row echelon form.

$$\begin{bmatrix} 1 & -1 & 1 & 0 & -4 \\ 1 & -1 & 2 & 2 & -5 \\ 3 & -3 & 2 & -2 & -11 \end{bmatrix} \xrightarrow[r_3-3r_1]{r_2-r_1} \begin{bmatrix} 1 & -1 & 1 & 0 & -4 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & -1 & -2 & 1 \end{bmatrix} \xrightarrow[r_3+r_2]{r_1-r_2} \begin{bmatrix} 1 & -1 & 0 & -2 & -3 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Writing the corresponding linear system of equations, we have

$$\begin{aligned} x_1 - x_2 - 2x_4 &= -3 \\ x_2 & \quad \text{free} \\ x_3 + 2x_4 &= -1 \\ x_4 & \quad \text{free} \end{aligned}$$

Here x_1 and x_3 are the basic variables, and x_2 and x_4 are free variables. Writing the general solution in vector form,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3 + x_2 + 2x_4 \\ x_2 \\ -1 - 2x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

□

4. (15 pts.) Let S be the following set of vectors in $\mathbb{R}^3$:

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix} \right\}$$

(a) Is $\mathbf{v} = \begin{bmatrix} 3 \\ 9 \\ 15 \end{bmatrix}$ in the span of S?

Solution. The vector $\mathbf{v}$ is in the span of S if and only if $A\mathbf{x} = \mathbf{v}$ is consistent. We test for consistency by finding the reduced row echelon form of the augmented matrix $[A\ \mathbf{v}]$.

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 3 & 4 & 9 \\ 4 & 7 & 2 & 15 \end{bmatrix} \xrightarrow[r_2-2r_1]{r_3-4r_1} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 6 & 3 \\ 0 & -1 & 6 & 3 \end{bmatrix}$$

$$\xrightarrow{-r_2} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -6 & -3 \\ 0 & -1 & 6 & 3 \end{bmatrix} \xrightarrow[r_1-2r_2]{r_3+r_2} \begin{bmatrix} 1 & 0 & 11 & 9 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \operatorname{rref} A$$

Since there is no row where the only nonzero entry is in the last column, the system $A\mathbf{x} = \mathbf{v}$ is consistent. Thus, $\mathbf{v}$ is in the span of S. □

(b) Is S linearly independent?

Solution. S is linearly independent if and only if the matrix equation $A\mathbf{x} = \mathbf{0}$ has only the zero solution (here the columns of A are the vectors in S). This occurs when the rank of A is equal to the number of columns of A, which in this case is 3. The reduced row echelon form of A can be obtained by ignoring the last column of the reduced row echelon form of $[A\ \mathbf{v}]$.

$$\operatorname{rref} A = \begin{bmatrix} 1 & 0 & 11 \\ 0 & 1 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

The rank of A is 2, which is less than the number of rows. Therefore, the columns of A are linearly dependent. □

5. (15 pts.) Let $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 5 \\ 1 & 3 & 1 \end{bmatrix}$.

(a) Compute A^{-1}.

Solution. We compute A^{-1} by putting the $n \times 2n$ matrix $[A\ I_{n\times n}]$ into the reduced row echelon form $[R\ B]$. If $R = I_{n\times n}$, then $B = A^{-1}$.

$$[A\ I] = \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 2 & 5 & 5 & 0 & 1 & 0 \\ 1 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[r_2-2r_1]{r_3-r_1} \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-r_2]{-r_3} \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right] \xrightarrow[r_1-2r_3]{r_2+r_3} \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & 3 & -1 & -1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right]$$

$$\xrightarrow{r_1-3r_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -10 & 3 & 5 \\ 0 & 1 & 0 & 3 & -1 & -1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right] = [I\ A^{-1}]$$

Thus,

$$A^{-1} = \begin{bmatrix} -10 & 3 & 5 \\ 3 & -1 & -1 \\ 1 & 0 & -1 \end{bmatrix}.$$

□

(b) Use A^{-1} to solve the matrix equation $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$.

Solution. Multiplying both sides on the left by A^{-1}, we have that

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -10 & 3 & 5 \\ 3 & -1 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}.$$

□

6. (20 pts.) Prove the following statements.

(a) Prove that if $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent, then so is $\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2\}$.

Solution. We consider a linear combination of the vectors in the second set:

$$d_1(\mathbf{v}_1 + \mathbf{v}_2) + d_2(\mathbf{v}_1 - \mathbf{v}_2) = \mathbf{0}$$

Rearranging, we have

$$(d_1 + d_2)\mathbf{v}_1 + (d_1 - d_2)\mathbf{v}_2 = \mathbf{0}$$

Since $\{v_1, v_2\}$ is linearly independent, the only coefficients satisfying the above linear combination are

$$\begin{aligned} d_1 + d_2 &= 0 \\ d_1 - d_2 &= 0 \end{aligned}$$

Solving this system of linear equations, the only solution is $d_1 = d_2 = 0$. Hence $\{\mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_2\}$ is linearly independent. □

(b) Let U be an $m \times n$ matrix, and Q an $n \times n$ matrix. Let $V = UQ$. Prove that span $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a subset of span $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$, where $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are the columns of V and $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ are the columns of U.

Solution. From the definition of matrix multiplication, $\mathbf{v}_i = U\mathbf{q}_i$ for $i = 1, \ldots, n$. Thus, each $\mathbf{v}_i$ is a linear combination of the columns of U, and so each $\mathbf{v}_i$ is in span $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$. Since $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a finite subset of span $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$, span $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a subset of span $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$. $\square$