Math 314 Fall 2007 **Notes for Final Exam**

Rotation Matrices:

$$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Matrix Algebra:
$(AB)^T = B^T A^T$
$A(c_1\mathbf{u}_1 + c_2\mathbf{u}_2) = c_1 A\mathbf{u}_1 + c_2 A\mathbf{u}_2$
$(AB)^{-1} = B^{-1}A^{-1}$
$(A^T)^{-1} = (A^{-1})^T$

Algorithm to find A^{-1}: $[A\ I] \xrightarrow{\text{RREF}} [I\ A^{-1}]$.

If the RREF of an augmented matrix $[A\ \mathbf{b}]$ contains a row where the only nonzero entry is in the last column, then $A\mathbf{x} = \mathbf{b}$ is inconsistent.

The rank of A is the number of nonzero rows in the RREF of A. The nullity of $A_{m\times n}$ is $n - \text{rank } A$.

$A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of A.

Performing an elementary row operation on A is the same as multiplying A by the corresponding elementary matrix E.

If R is the RREF of $A_{m\times n}$, then there exists an invertible matrix $P_{m\times m}$ such that $PA = R$.

Any linear relationship of the columns of A also applies to the columns of RREF A, and vice versa.

subspace from $A_{n\times n}$	dimension	basis
colspace A	rank(A)	cols of A corr to cols of RREF of A with leading 1s
rowspace A	rank(A)	nonzero rows of RREF of A
nullspace A	nullity(A)	vectors in parametric solution of $A\mathbf{x} = \mathbf{0}$

$A_{n\times n}$ is invertible $\Leftrightarrow$ rank $A = n$ $\Leftrightarrow$ RREF of A is I $\Leftrightarrow$ rank$(A) = n$ $\Leftrightarrow$ nullity$(A) = 0$ $\Leftrightarrow$ rows of A are linearly indep $\Leftrightarrow$ cols of A are lin indep $\Leftrightarrow$ rows of A span $\mathbb{R}^n$ $\Leftrightarrow$ cols of A span $\mathbb{R}^n$ $\Leftrightarrow$ rows of A form a basis for $\mathbb{R}^n$ $\Leftrightarrow$ cols of A form a basis for $\mathbb{R}^n$ $\Leftrightarrow$ $\det(A) \neq 0$ $\Leftrightarrow$ 0 is not an eigenvalue of A

$\det(AB) = (\det A)(\det B)$

Cramer's Rule: If $A_{n\times n}$ is invertible, then the unique solution to $A\mathbf{x} = \mathbf{b}$ is given by $x_i = \dfrac{\det(A_i(\mathbf{b}))}{\det(A)}$.

If $A_{n\times n}$ is invertible, then $A^{-1} = \frac{1}{\det(A)}\text{adj}(A)$, where the i, j entry of adj(A) is $(-1)^{i+j}\det(A_{ji})$.

λ is an eigenvalue of $A_{n\times n}$ if there is a nonzero vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$.

The characteristic polynomial of A is $\det(A - \lambda I)$.

The geometric multiplicity of an eigenvalue λ is the dimension of the eigenspace corresponding to λ. The geometric multiplicity is at most the algebraic multiplicity of λ.

$A_{n\times n}$ is diagonalizable if there exists a basis for $\mathbb{R}^n$ consisting of eigenvectors of A. Then $A = PDP^{-1}$, where P has the eigenvectors as columns and D is a diagonal matrix with the corresponding eigenvalues along the diagonal.

Quadratic Formula: If $at^2 + bt + c = 0$, then $t = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$. If $b^2 - 4ac < 0$, then the roots of the polynomial are complex.

If A is the transition matrix for a Markov chain, then 1 is an eigenvalue of A. The stationary distribution or steady-state vector is the probability vector $\mathbf{p}$ that is also an eigenvector corresponding to 1. The limit of $A^m\mathbf{v}$ as $m \to \infty$ and for any probability vector $\mathbf{v}$ is $\mathbf{p}$.

Let $\mathcal{B} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ and $\mathcal{C} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be two bases for $\mathbb{R}^n$. Let B be the matrix whose columns are $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ and C the matrix whose columns are $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$. For $\mathbf{v} \in \mathbb{R}^n$, $[\mathbf{v}]_{\mathcal{B}}$ is the coordinate vector of $\mathbf{v}$ with respect to the basis $\mathcal{B}$. The change-of-basis matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ satisfies $[\mathbf{v}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{v}]_{\mathcal{B}}$.
$[C\ B] \xrightarrow{\text{RREF}} [I\ P_{\mathcal{C}\leftarrow\mathcal{B}}]$. $P_{\mathcal{C}\leftarrow\mathcal{B}}$ is invertible and $P^{-1}_{\mathcal{C}\leftarrow\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{C}}$.

$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T\mathbf{v} \qquad \mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$

Pythagorean Theorem: $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$ if and only if $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are orthogonal.

Orthogonal Projection of $\mathbf{v}$ onto $\mathbf{u}$: $\text{proj}_{\mathbf{u}}(\mathbf{v}) = \dfrac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u}$

Orthogonal Projection of $\mathbf{v}$ onto a subspace W with orthogonal basis $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$:

$$\text{proj}_W(\mathbf{v}) = \frac{\mathbf{v} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{v} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 + \ldots + \frac{\mathbf{v} \cdot \mathbf{u}_k}{\mathbf{u}_k \cdot \mathbf{u}_k}\mathbf{u}_k$$

Gram-Schmidt Process: Start with a linearly indepedent set $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\} \subseteq \mathbb{R}^n$. Then $S' = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthogonal set where span S' = span S.

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{u}_1 \\
\mathbf{v}_2 &= \text{perp}_{\text{span}\{\mathbf{u}_1\}}\mathbf{v}_2 = \mathbf{u}_2 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 \\
&\vdots \\
\mathbf{v}_k &= \text{perp}_{\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{k-1}\}}\mathbf{v}_2 = \mathbf{u}_k - \frac{\mathbf{u}_k \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \ldots - \frac{\mathbf{u}_k \cdot \mathbf{v}_{k-1}}{\mathbf{v}_{k-1} \cdot \mathbf{v}_{k-1}}\mathbf{v}_{k-1}
\end{aligned}$$

If an orthonormal set is desired, then scale each vector by its length to produce a unit vector.

A matrix $Q_{n\times n}$ is orthogonal if the columns of Q form an orthonormal basis. Then $Q^TQ = I$.

The orthogonal complement $S^\perp$ is the set of all vectors in $\mathbb{R}^n$ that are orthogonal to each vector in a nonempty set $S \subseteq \mathbb{R}^n$. $S^\perp$ is a subspace. If W is a subspace of $\mathbb{R}^n$, then $\dim W + \dim W^\perp = n$.
$(\text{row}(A))^\perp = \text{null}(A)$ and $(\text{col}(A))^\perp = \text{null}(A^T)$.

Closest Vector Property: The closest vector in a subspace $W \subseteq \mathbb{R}^n$ to $\mathbf{v}$ is the orthogonal projection of $\mathbf{v}$ onto W.

$A_{n\times n}$ is orthogonally diagonalizable if $D = Q^TAQ$, where D is diagonal and Q is orthogonal.

Spectral Theorem: If $A_{n\times n}$ is a real matrix, then A is orthogonally diagonalizable if and only if A is symmetric.

Method of Least Squares: To find the line $y = a_0 + a_1x$ that best fits the data, let

$$A = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \\ 1 & x_n \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Then $\mathbf{x}$ is the solution to the normal equations $A^TA\mathbf{x} = A^T\mathbf{b}$.

The orthogonal projection matrix P_W can be found by computing $A(A^TA)^{-1}A^T$, where the columns of A are a basis for the subspace W.

In the space $\mathcal{C}[a, b]$ of continuous functions on a closed interval $[a, b]$, the inner product is

$$\langle f, g\rangle = \int_a^b f(x)g(x)\,dx.$$