

Rotation Matrices:

$$A_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

Matrix Algebra:

$$(AB)^T = B^T A^T$$

$$A(c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2) = c_1 A\mathbf{u}_1 + c_2 A\mathbf{u}_2$$

$$(AB)^{-1} = B^{-1} A^{-1}$$

$$(A^T)^{-1} = (A^{-1})^T$$

Algorithm to find  $A^{-1}$ :  $[A \ I] \xrightarrow{\text{RREF}} [I \ A^{-1}]$ .

If the RREF of an augmented matrix  $[A \ \mathbf{b}]$  contains a row where the only nonzero entry is in the last column, then  $A\mathbf{x} = \mathbf{b}$  is inconsistent.

The rank of  $A$  is the number of nonzero rows in the RREF of  $A$ . The nullity of  $A_{m \times n}$  is  $n - \text{rank } A$ .

$A\mathbf{x} = \mathbf{b}$  is consistent if and only if  $\mathbf{b}$  is in the column space of  $A$ .

Performing an elementary row operation on  $A$  is the same as multiplying  $A$  by the corresponding elementary matrix  $E$ .

If  $R$  is the RREF of  $A_{m \times n}$ , then there exists an invertible matrix  $P_{m \times m}$  such that  $PA = R$ .

Any linear relationship of the columns of  $A$  also applies to the columns of RREF  $A$ , and vice versa.

subspace from $A_{n \times n}$	dimension	basis
colspace $A$	$\text{rank}(A)$	cols of $A$ corr to cols of RREF of $A$ with leading 1s
rowspace $A$	$\text{rank}(A)$	nonzero rows of RREF of $A$
nullspace $A$	nullity( $A$ )	vectors in parametric solution of $A\mathbf{x} = \mathbf{0}$

$A_{n \times n}$  is invertible  $\Leftrightarrow \text{rank } A = n \Leftrightarrow \text{RREF of } A \text{ is } I \Leftrightarrow \text{rank}(A) = n \Leftrightarrow \text{nullity}(A) = 0 \Leftrightarrow \text{rows of } A \text{ are linearly indep} \Leftrightarrow \text{cols of } A \text{ are lin indep} \Leftrightarrow \text{rows of } A \text{ span } \mathbb{R}^n \Leftrightarrow \text{cols of } A \text{ span } \mathbb{R}^n \Leftrightarrow \text{rows of } A \text{ form a basis for } \mathbb{R}^n \Leftrightarrow \text{cols of } A \text{ form a basis for } \mathbb{R}^n \Leftrightarrow \det(A) \neq 0 \Leftrightarrow 0 \text{ is not an eigenvalue of } A$

$$\det(AB) = (\det A)(\det B)$$

Cramer's Rule: If  $A_{n \times n}$  is invertible, then the unique solution to  $A\mathbf{x} = \mathbf{b}$  is given by  $x_i = \frac{\det(A_i(\mathbf{b}))}{\det(A)}$ .

If  $A_{n \times n}$  is invertible, then  $A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$ , where the  $i, j$  entry of  $\text{adj}(A)$  is  $(-1)^{i+j} \det(A_{ji})$ .

$\lambda$  is an eigenvalue of  $A_{n \times n}$  if there is a nonzero vector  $\mathbf{v}$  such that  $A\mathbf{v} = \lambda\mathbf{v}$ .

The characteristic polynomial of  $A$  is  $\det(A - \lambda I)$ .

The geometric multiplicity of an eigenvalue  $\lambda$  is the dimension of the eigenspace corresponding to  $\lambda$ . The geometric multiplicity is at most the algebraic multiplicity of  $\lambda$ .

$A_{n \times n}$  is diagonalizable if there exists a basis for  $\mathbb{R}^n$  consisting of eigenvectors of  $A$ . Then  $A = PDP^{-1}$ , where  $P$  has the eigenvectors as columns and  $D$  is a diagonal matrix with the corresponding eigenvalues along the diagonal.

Quadratic Formula: If  $at^2 + bt + c = 0$ , then  $t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ . If  $b^2 - 4ac < 0$ , then the roots of the polynomial are complex.

If  $A$  is the transition matrix for a Markov chain, then 1 is an eigenvalue of  $A$ . The stationary distribution or steady-state vector is the probability vector  $\mathbf{p}$  that is also an eigenvector corresponding to 1. The limit of  $A^m \mathbf{v}$  as  $m \rightarrow \infty$  and for any probability vector  $\mathbf{v}$  is  $\mathbf{p}$ .

Let  $B = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$  and  $C = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$  be two bases for  $\mathbb{R}^n$ . Let  $B$  be the matrix whose columns are  $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$  and  $C$  the matrix whose columns are  $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ . For  $\mathbf{v} \in \mathbb{R}^n$ ,  $[\mathbf{v}]_B$  is the coordinate vector of  $\mathbf{v}$  with respect to the basis  $B$ . The change-of-basis matrix  $P_{C \leftarrow B}$  satisfies  $[\mathbf{v}]_C = P_{C \leftarrow B} [\mathbf{v}]_B$ .

$[C \ B] \xrightarrow{\text{RREF}} [I \ P_{C \leftarrow B}]$ .  $P_{C \leftarrow B}$  is invertible and  $P_{C \leftarrow B}^{-1} = P_{B \leftarrow C}$ .