

MATH 412, FALL 2006 - HOMEWORK 9

WARMUP PROBLEMS: Section 4.1 #4, 5. Section 4.2 #1, 2, 4. Do not write up!

EXTRA PROBLEMS: Section 4.1 #10, 14, 24(b), 25, 28, 31. Section 4.1 #19, 23, 25, 27, 28, 29 Section 4.2 #4, 9. Section 4.2 #11, 12, 14, 20, 23, 24 Do not write up!

WRITTEN PROBLEMS: Do five of the following six (all six if registered for four credits). Due FRIDAY, October 26.

TEST 2 is WEDNESDAY, Oct 25, 7-9pm in 341 Altgeld. Material covered is up through the lecture of Friday, Oct 20.

1. *Degree conditions for $\kappa' = \delta$.* Let G be a simple n -vertex graph. Use Corollary 4.1.13 to prove the following statements.

a) If $\delta(G) \geq \lfloor n/2 \rfloor$, then $\kappa'(G) = \delta(G)$. Prove this best possible by constructing for each $n \geq 4$ a simple n -vertex graph with $\delta(G) = \lfloor n/2 \rfloor - 1$ and $\kappa'(G) < \delta(G)$.

b) If $d(x) + d(y) \geq n - 1$ whenever $x \leftrightarrow y$, then $\kappa'(G) = \delta(G)$. Prove that this is best possible by constructing for each $n \geq 4$ and $\delta(G) = m \leq n/2 - 1$ an n -vertex graph G with $\kappa'(G) < \delta(G) = m$ in which $d(x) + d(y) \geq n - 2$ whenever $x \leftrightarrow y$.

2. Let F be a nonempty set of edges in G . Prove that F is an edge cut if and only if F contains an even number of edges from every cycle in G . For example, when $G = C_n$, every even subset of the edges is an edge cut, but no odd subset is an edge cut. (Hint: For sufficiency, the task is to show that the components of $G - F$ can be grouped into two nonempty collections so that every edge of F has an endpoint in each collection.)

3. Let G be a graph without isolated vertices. Prove that if G has no even cycles, then every block of G is an edge or an odd cycle.

4. Suppose that $\kappa(G) = k$ and $\text{diam } G = d$. Prove that $n(G) \geq k(d - 1) + 2$ and $\alpha(G) \geq \lceil (1 + d)/2 \rceil$. For each $k \geq 1$ and $d \geq 2$, construct a graph with connectivity k and diameter d for which equality holds in both bounds.

5. Let X and Y be disjoint sets of vertices in a k -connected graph G . Let $u(x)$ for $x \in X$ and $w(y)$ for $y \in Y$ be nonnegative integers such that $\sum_{x \in X} u(x) = \sum_{y \in Y} w(y) = k$. Prove that G has k pairwise internally disjoint X, Y -paths so that $u(x)$ of them start at x and $w(y)$ of them end at y , for $x \in X$ and $y \in Y$.

6. Let r, s , and t be vertices in a 2-connected graph G .

a) Prove that the vertices of G can be linearly ordered so that each vertex outside $\{s, t\}$ has a neighbor that is earlier in the order and a neighbor that is later in the order. (Hint: Use ear decompositions.)

b) Prove that G has two spanning trees T and T' such that for every $v \in V(G)$, the r, v -paths in T and T' have no common internal vertices. (Hint: Prove that there is a vertex numbering and trees T and T' such that in T the numbers increase along paths from r and in T' they decrease along paths from r , except for r itself.)