# MATH 412, FALL 2006 - HOMEWORK 7

WARMUP PROBLEMS: Section 3.1 #1, 4, 5, 6. Section 3.2 #2, 4. Do not write up!

EXTRA PROBLEMS: Section 3.1 #30, 36, 38, 40. Section 3.2 #6, 7, 9, 14. Do not write up!

WRITTEN PROBLEMS: Do five of the following six (all six if registered for four credits). Due Wednesday, October 11.

**1.** A *doubly stochastic matrix* $Q$ is a nonnegative real matrix in which every row and every column sums to 1. Prove that a doubly stochastic matrix $Q$ can be expressed $Q = c_1P_1 + \cdots + c_mP_m$, where $c_1, \ldots, c_m$ are nonnegative real numbers summing to 1 and $P_1, \ldots, P_m$ are permutation matrices. For example,

$$\begin{pmatrix} 1/2 & 1/3 & 1/6 \\ 0 & 1/6 & 5/6 \\ 1/2 & 1/2 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

(Hint: Use induction on the number of nonzero entries in $Q$.)

**2.** Let $G$ be an $X, Y$-bigraph with no isolated vertices, and define *deficiency* as in Exercise 3.1.32. Prove that Hall's Condition holds for a matching saturating $X$ if and only if each subset of $Y$ has deficiency at most $|Y| - |X|$.

**3.** Let $G$ be a nontrivial simple graph. Prove that $\alpha(G) \le n(G) - e(G)/\Delta(G)$. Conclude that $\alpha(G) \le n(G)/2$ when $G$ also is regular.

**4.** An algorithm to greedily build a large independent set $S$ iteratively selects a vertex of minimum degree in the remaining graph, adds it to $S$, and deletes it and its neighbors from the graph. Prove that this algorithm produces an independent set of size at least $\sum_{v \in V(G)} \frac{1}{d_G(v)+1}$ in a simple graph $G$.

**5.** Find a transversal of maximum total sum (weight) in each matrix below. Prove that there is no larger weight transversal by exhibiting a solution to the dual problem. Explain why this proves that there is no larger transversal.

$$\begin{pmatrix} 7 & 8 & 9 & 8 & 7 \\ 8 & 7 & 6 & 7 & 6 \\ 9 & 6 & 5 & 4 & 6 \\ 8 & 5 & 7 & 6 & 4 \\ 7 & 6 & 5 & 5 & 5 \end{pmatrix}$$

**6.** Prove that if man $x$ is paired with woman $a$ in some stable matching, then $a$ does not reject $x$ when the Gale–Shapley Proposal Algorithm is run with men proposing. Conclude that among all stable matchings, *every* man is as happy in the matching produced by this algorithm as in any stable matching. (Hint: Consider the first occurrence of such a rejection.)