MATH 412, FALL 2006 - HOMEWORK 4

WARMUP PROBLEMS: Section 1.3 #8, 46. Section 1.4 #1, 3, 4, 5, 8, 10. Section 2.1 #1, 4, 6, 7, 9. Do not write these up!

OTHERS OF INTEREST: Section 1.3 #57, 61. Section 1.4 #11, 14, 23, 29, 37. Section 2.1 #10, 18, 26, 27, 33, 34, 36, 37, 40. Do not write these up!

WRITTEN PROBLEMS: Solve and write up five of the following six (students registered for four credits or honors do all six problems). Due Wednesday, September 20.

1. Let $d_1, \dots, d_n$ be integers such that $d_1 \geq \cdots \geq d_n \geq 0$. Prove that there is a loopless graph (multiple edges allowed) with degree sequence $d_1, \dots, d_n$ if and only if $\sum d_i$ is even and $d_1 \leq d_2 + \cdots + d_n$.

2. Let $d_1 \leq \cdots \leq d_n$ be the vertex degrees of a simple graph G. Prove that G is connected if $d_j \geq j$ when $j \leq n - 1 - d_n$. (Hint: Consider a component that omits some vertex of maximum degree.)

3. *Orientations and P_3-decomposition.*

a) Prove that every connected graph has an orientation in which the number of vertices with odd outdegree is at most 1.

b) Use part (a) to conclude that a simple connected graph with an even number of edges can be decomposed into paths with two edges.

4. By Proposition 1.4.30, every tournament has a king. Let T be a tournament having no vertex with indegree 0.

a) Prove that if x is a king in T, then T has another king in $N^-(x)$.

b) Use part (a) to prove that T has at least three kings.

c) For each $n \geq 3$, construct an n-vertex tournament with exactly three kings.

(Comment: There exists an n-vertex tournament having exactly k kings whenever $n \geq k \geq 1$ except when $k = 2$ and when $n = k = 4$.)

5. Let G be an n-vertex simple graph having a decomposition into k spanning trees. Suppose also that $\Delta(G) = \delta(G) + 1$. Prove that $2k < n$, and determine the degree list of G in terms of n and k.

6. Let T be a tree. Prove that the vertices of T all have odd degree if and only if for all $e \in E(T)$, both components of $T - e$ have odd order.