MATH 412, FALL 2006 - HOMEWORK 3

WARMUP PROBLEMS: Section 1.3 #1, 2, 3, 5, 7. Do not write these up! Use these to clarify your understanding.

SUGGESTED PROBLEMS: Section 1.3 #32, 41, 45, 46, 49. Do not write these up! Think about some if you have time.

WRITTEN PROBLEMS: Solve and write up five of the following six (students registered for four credits or honors do all six problems). Due Wednesday, September 13.

1. For each positive integer k, determine whether there exists a simple graph having exactly i vertices of degree i for all $i \in [k]$ and no other vertices. (Hint: You must eliminate the impossible values and construct examples for the possible values. In the construction, start with disjoint complete graphs of various sizes.)

2. Let G be an n-vertex simple graph, where $n \geq 2$. Determine the maximum possible number of edges in G under each of the following conditions.

a) G has an independent set of size a.

b) G has exactly k components.

c) G is disconnected.

3. Let G be a loopless graph with average vertex degree $a = 2e(G)/n(G)$.

a) Prove that $G - x$ has average degree at least a if and only if $d(x) \leq a/2$.

b) Use part (a) to give an algorithmic proof that if $a > 0$, then G has a subgraph with minimum degree greater than $a/2$.

c) Show that there is no constant c greater than $1/2$ such that G must have a subgraph with minimum degree greater than ca; this proves that the bound in part (b) is best possible. (Hint: Use $K_{1,n-1}$.)

4. *Large bipartite subgraphs.*

a) Use induction on $n(G)$ to prove that every nontrivial loopless graph G has a bipartite subgraph H such that H has *more* than $e(G)/2$ edges.

b) Construct graphs $G_1, G_2, \ldots$, with G_n having $2n$ vertices, such that $\lim_{n\to\infty} f_n = 1/2$, where f_n is the fraction of $E(G_n)$ belonging to the largest bipartite subgraph of G_n.

5. Each game of *bridge* involves four players in two teams. Consider a club where four players can play a game only if no two of them have previously been partners that night. Suppose that 15 members arrive, but one decides to study graph theory. The other 14 play until each person has played five times. Prove that if the graph theorist now agrees to play, then at least one more game can be played.

6. *Stronger version of Mantel's Theorem.*

a) For a vertex v in a graph G, let $f(v)$ be the maximum size of an independent set contained in $N(v)$. Prove that $\sum_{v \in V(G)} f(v) \leq \lfloor n(G)^2/2 \rfloor$, and determine which graphs achieve equality. (Hint: Consider a maximum independent set in G.)

b) Use part (a) to obtain Theorem 1.3.23 (Mantel's Theorem).