MATH 412, FALL 2006 - HOMEWORK 12

WARMUP PROBLEMS: Section 5.2 #1, 2, 3, 5.

EXTRA PROBLEMS: Section 5.2 #7, 9, 14, 15, 21, 25, 29, 32, 40, 41.

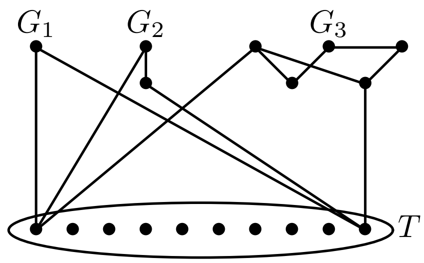
WRITTEN PROBLEMS: Do five of the following six (all six if registered for four credits). Due Wednesday, November 15.

1. Prove that if G has no induced $2K_2$, then $\chi(G) \le \binom{\omega(G)+1}{2}$. (Hint: Use a maximum clique to define a collection of $\binom{\omega(G)}{2} + \omega(G)$ independent sets that cover the vertices.

2. Let $G_1 = K_1$. For $k > 1$, construct G_k as follows. To the disjoint union $G_1 + \cdots + G_{k-1}$, and add an independent set T of size $\prod_{i=1}^{k-1} n(G_i)$. For each choice of $(v_1, \ldots, v_{k-1})$ in $V(G_1) \times \cdots \times V(G_{k-1})$, let one vertex of T have neighborhood $\{v_1, \ldots, v_{k-1}\}$. (In the sketch of G_4 below, neighbors are shown for only two elements of T.)

a) Prove that $\omega(G_k) = 2$ and $\chi(G_k) = k$.

b) Prove that G_k is k-critical.



3. Prove that every n-vertex simple graph with no $r+1$-clique has at most $(1-1/r)n^2/2$ edges. (Hint: This can be proved using Turán's Theorem or by induction on r without Turán's Theorem.)

4. Let G be a 4-critical graph having a separating set S of size 4. Prove that $G[S]$ has at most four edges.

5. *K_4-subdivisions.*

a) Prove that every simple graph with minimum degree at least 3 contains a K_4-subdivision. (Hint: Prove the stronger result that every nontrivial simple graph with at most one vertex of degree less than 3 contains a K_4-subdivision. The proof of Theorem 5.2.20 already shows that every 3-connected graph contains a K_4-subdivision.)

b) Conclude from part (a) that if $n(G) \ge 3$ and G does not contain a K_4-subdivision, then $e(G) \le 2n(G) - 3$. Give a construction to show that the bound is best possible.

6. A new type of combination lock uses card keys. The cards are numbered 1 through 100. The lock is set so that five of the cards are "valid". To open the lock, two valid cards must be inserted. Determine the minimum number of attempts (inserting a pair) that guarantees opening the lock.