MATH 412, FALL 2006 - HOMEWORK 11

WARMUP PROBLEMS: Section 5.1 #1, 3, 4, 5, 7, 9, 12, 14. Do not write up!

EXTRA PROBLEMS: Section 5.1 #21, 22, 31, 33, 35, 38, 39, 47, 48, 50, 51. Do not write up!

WRITTEN PROBLEMS: Do five of the following six (all six if registered for four credits). Due Wednesday, November 8.

1. Let G be a graph whose odd cycles are pairwise intersecting, meaning that every two odd cycles in G have a common vertex. Prove that $\chi(G) \leq 5$. Construct a graph to show that the bound cannot be improved.

2. For all $k \in \mathbb{N}$, prove that a graph G is 2^k-colorable if and only if G is the union of k bipartite graphs. (Hint: This generalizes Theorem 1.2.23.)

3. For all $k \in \mathbb{N}$, construct a tree T_k with maximum degree k and an ordering σ of $V(T_k)$ such that greedy coloring relative to the ordering σ uses $k+1$ colors. (Hint: Use induction and construct the tree and ordering simultaneously. Comment: This result shows that the performance ratio of greedy coloring to optimal coloring can be as bad as $(\Delta(G)+1)/2$.)

4. Prove that $\chi(G) + \chi(\overline{G}) \leq n(G) + 1$. (Hint: Use induction on $n(G)$.)

5. *Looseness of* $\chi(G) \geq n(G)/\alpha(G)$. Let G be an n-vertex graph, and let $c = (n+1)/\alpha(G)$. Use the preceding problem to prove that $\chi(G) \cdot \chi(\overline{G}) \leq (n+1)^2/4$, and use this to prove that $\chi(G) \leq c(n+1)/4$. For each odd n, construct a graph such that $\chi(G) = c(n+1)/4$.

6. Let G be a connected k-chromatic graph that is not a complete graph or a cycle whose length is an odd multiple of 3. Prove that under any optimal coloring, G has two vertices of the same color with a common neighbor. (Hint: Apply Brooks' Theorem.)