MATH 412, SPRING 2005 - HOMEWORK 6

WARMUP PROBLEMS: Section 3.1 #1, 4, 6, 7. Section 3.2 #2, 4. Do not write these up!

OTHER INTERESTING PROBLEMS: Section 3.1 #28, 30, 36, 38, 40. Section 3.2 #6, 7, 9. Do not write these up!

WRITTEN PROBLEMS: Solve and write five of the following six (all six if registered for four credits). Due Wednesday, March 2.

1. *Two applications of one idea.*

a) Use the König–Egerváry Theorem to prove that every bipartite graph G has a matching of size at least $e(G)/\Delta(G)$. Use this to conclude that every subgraph of $K_{n,n}$ with more than $(k-1)n$ edges has a matching of size at least k.

b) Let G be a nontrivial simple graph. Prove that $\alpha(G) \leq n(G) - e(G)/\Delta(G)$. Conclude that $\alpha(G) \leq n(G)/2$ when G also is regular.

2. Use the König–Egerváry Theorem to prove Hall's Theorem.

3. In an X,Y-bigraph G, the *deficiency* of a set S is $\operatorname{def}(S) = |S| - |N(S)|$; note that $\operatorname{def}(\varnothing) = 0$. Prove that $\alpha'(G) = |X| - \max_{S \subseteq X} \operatorname{def}(S)$. (Hint: Form a bipartite graph G' such that G' has a matching that saturates X if and only if G has a matching of the desired size, and prove that G' satisfies Hall's Condition.)

4. An algorithm to greedily build a large independent set S iteratively selects a vertex of minimum degree in the remaining graph, adds it to S, and deletes it and its neighbors from the graph. Prove that this algorithm produces an independent set of size at least $\sum_{v \in V(G)} \frac{1}{d_G(v)+1}$ in a simple graph G.

5. A *transversal* in a matrix is a set of entries having one in each row and in each column; its *weight* is the sum of the entries. Find a transversal of maximum weight in the matrix below. Prove that there is no larger weight transversal by exhibiting a solution to the dual problem. Explain why this proves that there is no larger transversal.

$$\begin{matrix} 4 & 4 & 4 & 3 & 6 \\ 1 & 1 & 4 & 3 & 4 \\ 1 & 4 & 5 & 3 & 5 \\ 5 & 6 & 4 & 7 & 9 \\ 5 & 3 & 6 & 8 & 3 \end{matrix}$$

6. Prove that if man x is paired with woman a in some stable matching, then a does not reject x when the Gale–Shapley Proposal Algorithm is run with men proposing. Conclude that among all stable matchings, *every* man is as happy in the matching produced by this algorithm as in any stable matching. (Hint: Consider the first occurrence of such a rejection.)