MATH 412, SPRING 2005 - HOMEWORK 5

TEST 1: Monday, February 21, 6-8PM in 145 Altgeld or 7-9PM in 143 Altgeld.

WARMUP PROBLEMS: Section 2.2 #3. Section 2.3 #1, 2, 3, 5, 22, 24. Section 3.1 #2, 3, 11. Do not write these up!

OTHER INTERESTING PROBLEMS: Section 2.2 #17, 20, 25, 28, 33. Section 2.3 #6, 7, 10, 11, 15, 20, 21. Section 3.1 #8, 9, 10, 15, 21, 25, 26. Do not write these up!

WRITTEN PROBLEMS: Solve and write five of the following six (all six if registered for four credits). Due on **Friday**, February 25 (due to the **test** on Monday evening (no Monday problem session). Problem session Tuesday afternoon as usual; problem session on Wednesday afternoon or evening if students want it. Solution set distributed Friday as usual.

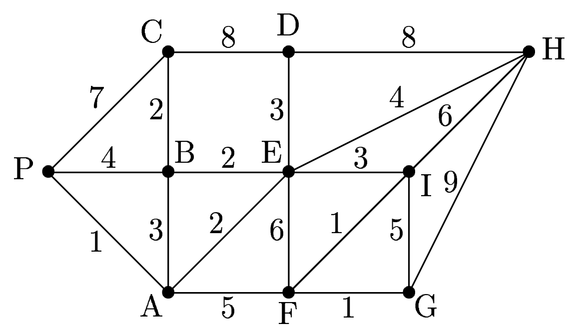
1. Prove that if the Graceful Tree Conjecture is true and T is a tree with m edges, then K_{2m} decomposes into $2m-1$ copies of T. (Hint: If every tree T with $m-1$ edges is graceful, then K_{2m-1} has a cyclically invariant decomposition into copies of T as in the proof of Theorem 2.2.16.)

2. Let G be a connected weighted graph.

a) Let T and T' be spanning trees of G. For $e \in E(T) - E(T')$, prove that there is an edge $e' \in E(T') - E(T)$ such that $T' + e - e'$ and $T - e + e'$ are both spanning trees of G.

b) Let T be a minimum-weight spanning tree in G, and let T' be another spanning tree in G. Prove that T' can be transformed into T by a list of steps that exchange one edge of T' for one edge of T, such that the edge set is always a spanning tree and the total weight never increases.

3. Every morning the Lazy Postman takes the bus to the Post Office (P). From there, he wants to reach home (H) as quickly as possible (**NOT** ending at the Post Office). Below is a map of the streets along which he must deliver mail, giving the number of minutes required to walk each block whether delivering or not. What must the edges traveled more than once satisfy? How many times will each edge be traversed in an optimal route?



4. Two people play a game on a graph G, alternately picking vertices. Player 1 starts at any vertex. Each subsequent choice must be adjacent to the preceding choice (of the other player) and not used before. Thus together they follow a path. The last player who moves wins.

Prove that the second player has a winning strategy if G has a perfect matching, and otherwise the first player has a winning strategy. (Hint: Be careful about the second part!)

5. The people in a club are planning their summer vacations. Trips $t_1, \ldots, t_n$ are available, but trip t_i has capacity c_i. Each person likes some of the trips and will travel on at most one. In terms of which people like which trips, derive a necessary and sufficient condition for being able to fill all trips (to capacity) with people who like them.

6. Prove that if $\frac{e(H)}{n(H)} \leq d$ for every subgraph H of a graph G, then G has an orientation with outdegree at most d at each vertex. (Hint: Apply Hall's Theorem to an X,Y-bigraph where $X = E(G)$ and Y consists of d copies of $V(G)$. Comment: A different method for an earlier problem!)