MATH 412, SPRING 2005 - HOMEWORK 4

TEST 1: Monday, February 21, 6-8PM in 145 Altgeld or 7-9PM in 143 Altgeld.

WARMUP PROBLEMS: Section 2.1 #1, 4, 6, 7, 9, 11, 13, 15, 16. Section 2.2 #1, 2. Do not write these up!

OTHER INTERESTING PROBLEMS: Section 2.1 #10, 18, 26, 27, 33, 34, 37, 40, 44, 47, 59, 64. Section 2.2 #5, 7, 12. Do not write these up!

WRITTEN PROBLEMS: Solve and write up five of the following six (students registered for four credits or honors do all six problems). Due Wednesday, February 16.

1. Let G be an n-vertex simple graph having a decomposition into k spanning trees. Suppose also that $\Delta(G) = \delta(G) + 1$. For $2k \geq n$, show that this is impossible. For $2k < n$, determine the degree sequence of G in terms of n and k.

2. Let T be a tree of even order. Prove that T has exactly one spanning subgraph in which every vertex has odd degree.

3. Given $x \in V(G)$, let $s(x) = \sum_{v \in V(G)} d(x, v)$. The *barycenter* of G is the subgraph induced by the set of vertices minimizing $s(x)$ (the set is also called the *median*).

a) Prove that the barycenter of a tree is a single vertex or an edge. (Hint: Study $s(u) - s(v)$ when u and v are adjacent.)

b) Determine the maximum distance between the center and the barycenter in a tree of diameter d. (Example: In the tree of Figure 1 below, the center is the edge xy, the barycenter contains only z, and the distance between them is 1.)

4. Given a connected graph G, define a new graph G' having one vertex for each spanning tree of G, with vertices adjacent in G' if and only if the corresponding trees have exactly $n(G) - 2$ common edges. Prove that G' is connected. For $n \in \mathbb{N}$, determine the maximum value the diameter of G' can have when G is a connected graph with n vertices. For example, when G is the kite (a graph obtained by deleting one edge from K_4), the resulting graph G' is shown in Figure 2 below.

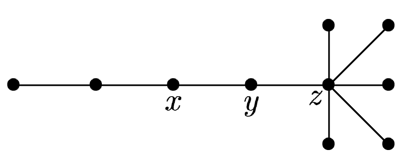

Figure 1

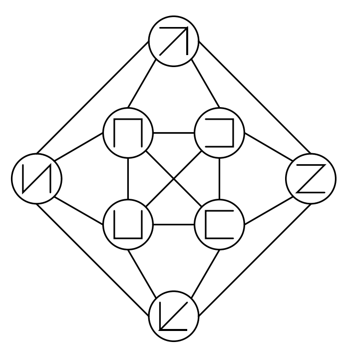

Figure 2

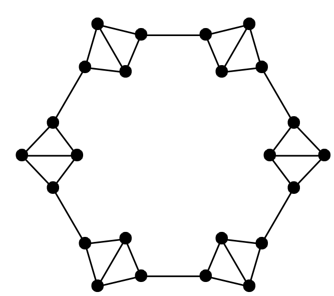

Figure 3

5. Let G be the 3-regular graph with $4m$ vertices formed from m pairwise disjoint kites by adding m edges to link them in a ring, as shown in Figure 3 above for $m = 6$. Prove that $\tau(G) = 2m8^m$.

6. Count the following sets of trees with vertex set $[n]$, giving two proofs for each: one using the Prüfer correspondence and one by direct counting arguments.

a) trees that have 2 leaves.

b) trees that have $n - 2$ leaves.