MATH 412, SPRING 2005 - HOMEWORK 3

WARMUP PROBLEMS: Section 1.3 #8, 46. Section 1.4 #1, 3, 4, 5, 8, 10. Do not write these up! Use these to clarify your understanding.

OTHER INTERESTING PROBLEMS: Section 1.3 #47, 49, 51, 52, 57, 61, 64. Section 1.4 #11, 14, 20, 21, 23, 25, 28, 29, 32, 36, 40. Do not write these up! Think about some if you have time.

WRITTEN PROBLEMS: Solve and write up five of the following six (students registered for four credits or honors do all six problems). Due Wednesday, February 9.

COMMENT: When using induction to prove an implication, keep the template of Remark 1.3.25 in mind. Induction is a promising technique when a current instance of the problem can be solved by using a solution to a smaller instance.

1. Let G be an n-vertex simple graph, where $n \geq 2$. Determine the maximum possible number of edges in G under each of the following conditions.

a) G has an independent set of size a.

b) G has exactly k components.

c) G is disconnected.

2. Let G be a loopless graph with average vertex degree $a = 2e(G)/n(G)$.

a) Prove that $G - x$ has average degree at least a if and only if $d(x) \leq a/2$.

b) Use part (a) to give an algorithmic proof that if $a > 0$, then G has a subgraph with minimum degree greater than $a/2$.

c) Show that there is no constant c greater than $1/2$ such that G must have a subgraph with minimum degree greater than ca; this proves that the bound in part (b) is best possible. (Hint: Use $K_{1,n-1}$.)

3. Each game of *bridge* involves four players split into two teams. A certain bridge club has a rule that four players cannot play a game if any two of them have previously been a team that night. Suppose that 15 members arrive, but one decides to study graph theory. The other 14 people play until *each* has played four times. Next the rules allow them to play six more games (12 partnerships). Prove that if the graph theorist now agrees to play, then at least one more game can be played.

4. Let $d_1, \ldots, d_n$ be integers such that $d_1 \geq \cdots \geq d_n \geq 0$. Prove that there is a loopless graph (multiple edges allowed) with degree sequence $d_1, \ldots, d_n$ if and only if $\sum d_i$ is even and $d_1 \leq d_2 + \cdots + d_n$.

5. Prove that if $e(H)/n(H) \leq d$ for every subgraph H of a graph G, then G has an orientation in which every vertex has outdegree at most d. (Hint: Iteratively modify an arbitrary orientation to obtain a desired orientation.)

6. Consider the following algorithm whose input is a tournament T.

1) Select a vertex x in T.

2) If x has indegree 0, call x a king of T and stop.

3) Otherwise, delete $\{x\} \cup N^+(x)$ from T to form T'.

4) Run the algorithm on T'; call the output a king in T and stop.

Prove that this algorithm terminates and produces a king in T.