MATH 412, SPRING 2005 - HOMEWORK 12

Test #3 on Monday, April 25: 6–8PM in ??? Altgeld Hall or 7–9PM in 143 Altgeld Hall, covering through Homework 12 (coloring of planar graphs, but not crossing number).

WARMUP PROBLEMS: Section 6.1 #9, 10, 11. Section 6.2 #1, 4. Section 6.3 #1. Do not write these up! Use them to check your understanding.

OTHERS OF INTEREST: Section 6.1: #25, 26, 27, 29, 30, 34, 36. Section 6.2: #6, 8, 11, 14. Section 6.3: #5, 11, 13, 14, 15. Do not write these up!

WRITTEN HOMEWORK: Do five of the following six problems (all six if registered for one unit). Due Wednesday, April 20.

1. Use Euler's formula to count the regions formed by n pairwise-crossing lines in the plane, where no three lines have a common point. (Comment: Proving the formula by induction is not acceptable. Hint: Modify the picture to obtain a finite plane graph.)

2. Give three proofs that the Petersen graph is nonplanar.

a) Using Kuratowski's Theorem.

b) Using Euler's Formula and the fact that the Petersen graph has girth 5.

c) Using the planarity-testing algorithm of Demoucron–Malgrange–Pertuiset.

3. Use Kuratowski's Theorem to prove that G is outerplanar if and only if it has no subgraph that is a subdivision of K_4 or $K_{2,3}$. (Hint: Apply Kuratowski's Theorem to a graph H constructed from G so that G is outerplanar if and only if H is planar. This is much easier than trying to mimic the proof of Kuratowski's Theorem.)

4. A graph G has a graph H as a *minor* if H can be obtained from G by performing deletions and/or contractions of edges. A graph G satisfies *Wagner's Condition* if neither K_5 nor $K_{3,3}$ is a minor of G.

a) Show that both deletion and contraction of edges preserve planarity. Conclude from this that Wagner's Condition is necessary for planarity.

b) Use Kuratowski's Theorem to prove that Wagner's Condition is also sufficient.

c) Wagner's Theorem is different from Kuratowski's: the Petersen graph has K_5 as a minor, but it contains no subdivision of K_5. Nevertheless, prove that a graph G has $K_{3,3}$ as a minor if and only if it contains a subdivision of $K_{3,3}$.

5. Without using the Four Color Theorem, prove that every planar graph with at most 12 vertices is 4-colorable and that every planar graph with at most 32 edges is 4-colorable.

6. *Coloring of outerplanar graphs.*

a) Give two proofs that every outerplanar graph is 3-colorable, one using the Four Color Theorem and one without it.

b) Apply part (a) to prove the Art Gallery Theorem: If an art gallery is laid out as a simple polygon with n sides, then it is possible to place $\lfloor n/3 \rfloor$ guards such that every point of the interior is visible to some guard. Construct a polygon that requires $\lfloor n/3 \rfloor$ guards.