MATH 412, SPRING 2005 - HOMEWORK 10

WARMUP PROBLEMS: Section 5.1 #16, 17, 19. Section 5.2 #1, 2, 3, 5.

OTHERS OF INTEREST: Section 5.1 #44, 48, 50, 51. Section 5.2 #7, 9, 11, 14, 15, 18, 21, 25, 26, 29, 32, 40, 41. Do not write these up!

WRITTEN HOMEWORK: Do five of the following six problems (all six if registered for one unit). Due Wednesday, April 6.

1. Let G be a connected k-chromatic graph that is not a complete graph or a cycle of length congruent to 3 modulo 6. Prove that every proper k-coloring of G has two vertices of the same color with a common neighbor. (Hint: Use Brooks' Theorem.)

2. *Paths and chromatic number in digraphs.*

a) Let $G = F \cup H$. Prove that $\chi(G) \le \chi(F)\chi(H)$.

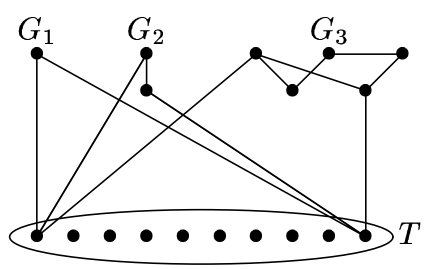
b) Consider an orientation D of G and a function $f\colon V(G) \to \mathbb{R}$. Use part (a) and Theorem 5.1.21 to prove that if $\chi(G) > rs$, then D has a path $u_0 \to \cdots \to u_r$ with $f(u_0) \le \cdots \le f(u_r)$ or a path $v_0 \to \cdots \to v_s$ with $f(v_0) > \cdots > f(v_s)$.

c) Use part (b) to prove that every list of $rs + 1$ distinct numbers has an increasing sublist of size $r + 1$ or a decreasing sublist of size $s + 1$.

3. Let $G_1 = K_1$. For $k > 1$, construct G_k as follows. To the disjoint union $G_1 + \cdots + G_{k-1}$, and add an independent set T of size $\prod_{i=1}^{k-1} n(G_i)$. For each choice of $(v_1, \ldots, v_{k-1})$ in $V(G_1) \times \cdots \times V(G_{k-1})$, let one vertex of T have neighborhood $\{v_1, \ldots, v_{k-1}\}$. (In the sketch of G_4 below, neighbors are shown for only two elements of T.)

a) Prove that $\omega(G_k) = 2$ and $\chi(G_k) = k$.

b) Prove that G_k is k-critical.



4. Prove that every n-vertex simple graph with no $r+1$-clique has at most $(1 - 1/r)n^2/2$ edges. (Hint: This can be proved using Turán's Theorem or by induction on r without Turán's Theorem.)

5. Let G be a 4-critical graph having a separating set S of size 4. Prove that $G[S]$ has at most four edges.

6. *K_4-subdivisions.*

a) Prove that every simple graph with minimum degree at least 3 contains a K_4-subdivision. (Hint: Prove the stronger result that every nontrivial simple graph with at most one vertex of degree less than 3 contains a K_4-subdivision. The proof of Theorem 5.2.20 already shows that every 3-connected graph contains a K_4-subdivision.)

b) Conclude from part (a) that if $n(G) \ge 3$ and G does not contain a K_4-subdivision, then $e(G) \le 2n(G) - 3$. Give a construction to show that the bound is best possible.