Math 413 Fall 2005 **Homework #10** due Wednesday, November 2, 2005

1. Problem §7.8, #42(b,c).
2. (a) Prove that if

$$a_n = \sum_{k=0}^{n} \binom{n}{k} b_{n-k}, \quad \text{for all } n \geq 0, \tag{$\dagger$}$$

then

$$g_a^{(e)}(x) = e^x g_b^{(e)}(x),$$

where $g_a^{(e)}(x)$ is the exponential generating function for a_n and $g_b^{(e)}(x)$ is the exponential generating function for b_n. (*Hint*: multiply the expression (†) by $\frac{x^n}{n!}$ and sum over all n.)

(b) Use part 2a to prove that the exponential generating function for D_n is

$$g^{(e)}(x) = \sum_{n=0}^{\infty} D_n \frac{x^n}{n!} = \frac{e^{-x}}{1-x}.$$

(*Hint*: use the fact that $n! = \sum_{k=0}^{n} \binom{n}{k} D_{n-k}$.)

3. Problem §8.6, #1. Prove bijectively.
4. Problem §8.6, #2. Prove using any method.
5. Let $f(n, \ell)$ be the number of lattice paths from $(0, 0)$ to $(n, n + \ell)$ that never pass *above* the line $y = x + \ell$ in the plane. (Remember that lattice paths take steps of length 1 either horizontally to the right or vertically up.) Prove that

$$f(n, \ell) = \frac{\ell + 1}{n + \ell + 1} \binom{2n + \ell}{n}.$$