**Math 413 Fall 2005 Extra Homework #3** (for students registered for 4.0 credits)

Do 6 of the following 8 problems. Additionally, write up two solutions in a form that can be distributed to the other students. Due Friday, December 2.

Read chapter 14 in the textbook on Pólya counting (section 14.1 is just a review of permutation and symmetry groups).

1. Problem 14.4.38 from the textbook.
2. Problem 14.4.42 from the textbook.
3. The Petersen graph $P$ is the graph whose vertices are the two elements subsets of $\{1, 2, 3, 4, 5\}$ and where two vertices are adjacent if and only if the corresponding subsets are disjoint.
   (a) Prove that the automorphism group of $P$ is isomorphic to $S_5$.
   (b) Determine the number of nonequivalent colorings of $P$ with 3 colors.
4. Let $\Gamma$ be your favorite non-cyclic group of order at least 6. Let $D$ be a discrete object such that $\Gamma$ is a symmetry group of $D$. (Note that $\Gamma$ does not have to be the full automorphism group of $D$.) Count the number of nonequivalent colorings of $D$ with 3 colors. (*Let's not be silly here: Don't pick a* $\Gamma$ *and* $D$ *from the textbook or from a previous exercise.*)
5. Identities for Stirling numbers.
   (a) Prove that $S(n+1, m+1) = \sum_k \binom{n}{k} S(k, m)$. (*Hint*: Give a bijective proof.)
   (b) Apply part 5a to prove that $\binom{n}{m} = \sum_k S(n+1, k+1)s(k, m)$. (*Hint*: Use #2 from Homework 10.)
6. Let $p_{n,k}$ be the number of partitions of $n$ into $k$ parts. Prove the following:
   (a) $p_{2r+k,r+k}$ is independent of $k$.
   (b) $p_{r+k,k}$ counts the partitions of $r$ into parts of size at most $k$.
   (c) $p_{r+k,k}$ counts the partitions $r + k(k+1)/2$ into $k$ distinct parts.
   (d) $p_{n,k} = p_{n-1,k-1} + p_{n-k,k}$.
7. For $n \geq 0$, let $a_n$ denote the number of congruence classes of triangles with integer-length sides and perimeter $n$. Equivalently, $a_n$ is the number of partitions of the integer $n$ into three parts satisfying the strict triangle inequality.
   (a) For $k \geq 0$, prove that $a_{2k}$ is the number of partitions of $k$ into three parts.
   (b) For $k \geq 3$, prove that $a_{2k-3} = a_{2k}$.

(c) Use the previous two parts and manipulation of generating functions to prove that the ordinary generating function for $a_n$ is $\frac{x^3}{(1-x^2)(1-x^3)(1-x^4)}$.

8. For $n > 0$, let $P(n) = \sum_{i=0}^{n-1} p(i)$, where $p(i)$ counts all partitions of $i$.

   (a) Consider a set $S$ of marbles consisting of one black marble and $n-1$ white marbles. White marbles are indistinguishable from one another. Prove that $P(n)$ is the number of distinguishable partitions of $S$.

   (b) Prove that $P(n)$ is the sum, over all partitions of $n$, of the number of distinct part-sizes in the partition. For example, $4$, $2+2$, $1+1+1+1$ use only one size, while $3+1$ and $2+1+1$ each use two, so the sum is 7 when $n = 4$.