Math 413 Fall 2005 Extra Homework #2 (for students registered for 4.0 credits)

Do 6 of the following 8 problems. Additionally, write up two solutions in a form that can be distributed to the other students. Due Friday, October 28.

Read section 6.6 in the textbook on Möbius inversion. Make sure that you read the errata for the book—otherwise, it's confusing in places.

1. Let n be a positive integer and let $p_1, p_2, \ldots, p_k$ be all the different prime numbers that divide n. Consider the Euler function ϕ defined by

$$\phi(n) = |\{k : 1 \leq k \leq n, \mathrm{GCD}\{k, n\} = 1\}|.$$

Use the inclusion-exclusion principle to show that

$$\phi(n) = n \prod_{i=1}^{k} (1 - \frac{1}{p_i}).$$

2. Consider the linearly ordered set $1 < 2 < \ldots < n$. Let $F\colon \{1, 2, \ldots, n\} \to \mathbb{R}$ be a function and let $G\colon \{1, 2, \ldots, n\} \to \mathbb{R}$ be defined by

$$G(m) = \sum_{k=1}^{m} F(k), \qquad (1 \leq k \leq n).$$

Apply Möbius inversion to get F in terms of G.

3. Consider the partially ordered set $(\mathcal{P}(X), \subseteq)$ of subsets of $X = \{1, 2, 3\}$ partially ordered by containment. Let a function f in $\mathcal{F}(\mathcal{P}(X))$ be defined by

$$f(A, B) = \begin{cases} 1, & \text{if } A = B, \\ 2, & \text{if } A \subset B \text{ and } |B| - |A| = 1, \\ 1, & \text{if } A \subset B \text{ and } |B| - |A| = 2, \\ -1, & \text{if } A \subset B \text{ and } |B| - |A| = 3. \end{cases}$$

Find the inverse of f with respect to the convolution product.

4. A proper coloring is an assignment of colors to the vertices of a graph G so that two adjacent vertices receive different colors. For a graph G and a set S of edges in G, let $c(S)$ denote the number of connected components in the subgraph of G with vertex set $V(G)$ and edge set S. Prove that the number $\chi(G; k)$ of proper colorings of G using colors chosen from $\{1, 2, \ldots, k\}$ is

$$\chi(G; k) = \sum_{S \subseteq E(G)} (-1)^{|S|} k^{c(S)}.$$

5. Let t_n denote the number of triangles in the equilateral triangular grid with side length n. Note that $t_1 = 1$, $t_2 = 5$, and $t_3 = 13$. Use inclusion-exclusion to derive a third-order recurrence for t_n. Solve it by using the characteristic equation method.

6. Let a_n be the number of ways to select $r \in \mathbb{N}$, roll one six-sided die r times in succession, and obtain a sum of n. Find a generating function for a_n, and express it as the ratio of two polynomials with at most three terms each.

7. Let $a_{n,k}$ be the number of ways to tile a 3-by-n rectangle using 1-by-1 squares and k copies of L, where L is a 2-by-2 square with the top right 1-by-1 corner missing (no rotation is allowed). Build a generating function in two variables for the numbers $a_{n,k}$.

8. Let p be a polynomial of degree r.

   (a) Prove that if $r < n$, then $\sum_{k=0}^{n}(-1)^k\binom{n}{k}p(k) = 0$.

   (b) For $m, n, r \in \mathbb{N}$, evaluate $\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{m-k}{r}$. (Treat $\binom{m-k}{r}$ as an extended binomial coefficient that is nonzero when $m - k$ is negative.)