Math 413 Fall 2005 Extra Homework #1 (for students registered for 4.0 credits)

Do 6 of the following 8 problems. Additionally, write up two solutions in a form that can be distributed to the other students. Due Friday, September 30.

1. Prove a more general form of Ramsey's Theorem: given k positive integers $a_1, a_2, \dots, a_k$, there exists a positive integer p such that

   $$K_p \to K_{a_1}, K_{a_2}, \dots, K_{a_k}.$$

   (*Note*: This means that for any coloring of the edges of K_p with k colors, a monochromatic K_{a_i} of color i can be found for some i.)

2. Prove that if $\binom{n}{k} \cdot 2^{1-\binom{k}{2}} < 1$, then $R(k,k) > n$. (*Hint*: Use the averaging principle.)

3. Given $k > 0$, prove that there exists a least integer s_k such that every k-coloring of the integers $1, \dots, s_k$ yields a monochromatic x, y, z (not necessarily distinct) such that $x + y = z$.

4. Let S be a set of $R(m, m; 3)$ points in the plane no three of which are collinear. Prove that S contains m points that form a convex m-gon. (*Note*: $R(m, m; 3)$ is the least number N of vertices such that in a 2-coloring of the *triples* of a vertex set of size N, there is a set T of size m such that all of the triples on T are the same color.)

5. Count the lists of m 1s and n 0s that have exactly k runs of 1s, where a *run* is a maximal set of consecutive entries with the same value.

6. For $m \in \mathbb{N}$, let $f(m) = \sum_{j=1}^{m}(m-j)2^{j-1}$. Obtain a simple formula for $f(m)$ inductively, and then give a direct combinatorial proof by counting a set in two ways.

7. For nonnegative integers m, n, r, prove the following identities:

   (a)
   $$\sum_k \binom{r}{m+k}\binom{s}{n-k} = \binom{r+s}{m+n},$$

   (b)
   $$\sum_k \binom{r}{m+k}\binom{s}{n+k} = \binom{r+s}{r-m+n}.$$

8. For nonnegative integers m, n, prove that

$$\sum_k \binom{m}{k}\binom{n+k}{m} = \sum_k \binom{n}{k}\binom{m}{k} 2^k.$$

(*Hint*: Give a combinatorial proof.)