Math 250:B1, Quiz #4 10 points Name ____________________
June 3, 2003 Show all work!

1. (2 pts.) State the condition for when a set $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ of vectors of $\mathbb{R}^n$ is *linearly dependent*. (*Hint*: Your answer should begin "The set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is linearly dependent when....")

   Solution: The set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is linearly dependent when there is a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ with coefficients not all equal to 0 that equals the zero vector; that is, when there are scalars $c_1, c_2, \ldots, c_k$, not all zero, such that

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_k\mathbf{u}_k = \mathbf{0}.$$

   Equivalently, if $A = [\mathbf{u}_1\ \mathbf{u}_2\ \ldots\ \mathbf{u}_k]$, then the columns of A are linearly dependent if and only if there is a nonzero solution to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

2. (8 pts.; §1.7, #11) Theorem 1.7 states (among other things) that the columns of $A_{m\times n}$ are linearly independent if and only if the only solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{0}$. Use this fact or any other method that you prefer to determine whether the set

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

   is linearly dependent or linearly independent.

   Solution:

   Let A be the matrix whose columns are the vectors given above:

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 2 \\ -2 & 1 & 1 \end{bmatrix}.$$

   The reduced row echelon form of the augmented matrix $[A\ \mathbf{0}]$ is

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

   Writing the general solution of $A\mathbf{x} = \mathbf{0}$, we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_3 \\ 3x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}.$$

   Since we have a free variable x_3, there are an infinite number of solutions to $A\mathbf{x} = \mathbf{0}$, and in particular there is a nonzero solution. Thus, by Theorem 1.7, the columns of A are linearly dependent.