Math 250:B1, Quiz #12 | 10 points | Name ________________________
July 1, 2003 | Show all work!

Let W be a subspace of $\mathbb{R}^n$, and let P_W be the orthogonal projection matrix of W.

from the formula sheet:
The orthogonal projection matrix P_W can be found by computing $C(C^TC)^{-1}C^T$, where the columns of C are a basis for the subspace W.

1. (5 pts.; §6.3, #27) Prove that $(P_W)^T = P_W$.

   Solution:

   Let C be a matrix whose columns are a basis for W. Then

$$\begin{aligned}
(P_W)^T &= [C(C^TC)^{-1}C^T]^T \\
&= (C^T)^T[(C^TC)^{-1}]^TC^T \\
&= C[(C^TC)^T]^{-1}C^T \\
&= C[C^T(C^T)^T]^{-1}C^T \\
&= C[C^TC]^{-1}C^T \\
&= P_W.
\end{aligned}$$

2. (5 pts.; §6.3, #31) Prove that $(P_W\mathbf{u})\cdot\mathbf{v} = \mathbf{u}\cdot(P_W\mathbf{v})$ for any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$. (*Hint*: Recall that the dot product can also be defined in terms of a matrix product.)

   Solution:

   Recall that the dot product $\mathbf{x}\cdot\mathbf{y}$ can also be written as the matrix product $\mathbf{x}^T\mathbf{y}$. Then

$$\begin{aligned}
(P_W\mathbf{u})\cdot\mathbf{v} &= (P_W\mathbf{u})^T\mathbf{v} \\
&= \mathbf{u}^T(P_W)^T\mathbf{v} \\
&= \mathbf{u}^TP_W\mathbf{v} \quad \text{from above} \\
&= \mathbf{u}\cdot(P_W\mathbf{v})
\end{aligned}$$

Alternate Solution:

By Theorem 6.7, $\mathbf{u}$ can be written uniquely as $\mathbf{w}_1 + \mathbf{z}_1$, where $\mathbf{w}_1 \in W$ and $\mathbf{z}_1 \in W^\perp$. Similarly, $\mathbf{v} = \mathbf{w}_2 + \mathbf{z}_2$, where $\mathbf{w}_2 \in W$ and $\mathbf{z}_2 \in W^\perp$. Then

$$\begin{aligned}(P_W\mathbf{u}) \cdot \mathbf{v} &= (P_W(\mathbf{w}_1 + \mathbf{z}_1)) \cdot \mathbf{v} \\ &= \mathbf{w}_1 \cdot \mathbf{v} \\ &= \mathbf{w}_1 \cdot (\mathbf{w}_2 + \mathbf{z}_2) \\ &= (\mathbf{w}_1 \cdot \mathbf{w}_2) + (\mathbf{w}_1 \cdot \mathbf{z}_2) \\ &= \mathbf{w}_1 \cdot \mathbf{w}_2.\end{aligned}$$

Likewise,

$$\begin{aligned}\mathbf{u} \cdot (P_W\mathbf{v}) &= \mathbf{u} \cdot (P_W(\mathbf{w}_2 + \mathbf{z}_2)) \\ &= \mathbf{u} \cdot (\mathbf{w}_2) \\ &= (\mathbf{w}_1 + \mathbf{z}_1) \cdot \mathbf{w}_2 \\ &= (\mathbf{w}_1 \cdot \mathbf{w}_2) + (\mathbf{z}_1 \cdot \mathbf{w}_2) \\ &= \mathbf{w}_1 \cdot \mathbf{w}_2.\end{aligned}$$

Thus, $(P_W\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (P_W\mathbf{v})$.