Math 250:B1, Midterm #2 100 points Name ________________________

June 24, 2003 *Show all your work. Full credit will not be given for an answer alone.*

1. (10 pts.) Let $A = \begin{bmatrix} 4 & 2 & 2 & -3 \\ 6 & -1 & 1 & 5 \\ 0 & -3 & 0 & 0 \\ 2 & -5 & 0 & 0 \end{bmatrix}$.

   (a) Compute $\det A$.

   Solution:

$$\begin{aligned}
\det A &= \det \begin{bmatrix} 4 & 2 & 2 & -3 \\ 6 & -1 & 1 & 5 \\ 0 & -3 & 0 & 0 \\ 2 & -5 & 0 & 0 \end{bmatrix} \\
&= -3 \cdot (-1)^{3+2} \det \begin{bmatrix} 4 & 2 & -3 \\ 6 & 1 & 5 \\ 2 & 0 & 0 \end{bmatrix} && \text{by expanding about the third row} \\
&= 3 \cdot 2 \cdot (-1)^{3+1} \det \begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix} && \text{by expanding about the third row} \\
&= 3 \cdot 2 \cdot (2 \cdot 5 - 1 \cdot (-3)) && \text{from our formula for det of } 2 \times 2 \text{ matrices} \\
&= 78.
\end{aligned}$$

   (b) Is A invertible? Why or why not?

   Solution:
   A is invertible because $\det A$ does not equal 0.

2. (24 pts.) Determine if each statement is true or false. Justify your answer if you claim the statement is true; if false, explain why the statement is false or provide an example demonstrating that it is false.

   (a) If B is the 4×4 matrix obtained from $A_{4\times 4}$ by adding 2 times the first row to the third row and then swapping the second and fourth rows, then $\det B = \det A$.

   Solution:

   False. Performing the elementary row operation on a matrix of adding a multiple of one row to another does not change the determinant, but swapping two rows multiplies the determinant by -1. Thus, $\det B = -\det A$. If A is an invertible 4×4 matrix (for instance, $I_{4\times 4}$), then $\det A \neq 0$, and A is a counterexample to the statement.

   (b) If λ is an eigenvalue of A, then λ^2 is an eigenvalue of A^2.

   Solution:

   True. Since λ is an eigenvalue of A, there is a nonzero eigenvector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$. Then

$$A^2\mathbf{v} = A(A\mathbf{v}) = A(\lambda\mathbf{v}) = \lambda A\mathbf{v} = \lambda^2\mathbf{v}.$$

   Thus, λ^2 is an eigenvalue of A^2.

   (c) Let A be a 5×7 matrix that is not the zero matrix. The dimension of Null A^T can be 5.

   Solution:

   False. The dimension of Null A^T is nullity A^T. But then

$$\begin{aligned} \text{nullity } A^T &= (\#\text{ of cols of } A^T) - \text{rank } A^T \\ &= 5 - \text{rank } A^T \\ &= 5 - \text{rank } A^T. \end{aligned}$$

   But since A is a nonzero matrix, the reduced row echelon form of A has a nonzero row, and hence rank $A < (\#\text{ of rows of } A) = 5$. Thus, nullity $A^T < 5$.

(d) For any $n \times n$ matrices A and B, $\det(A+B) = \det A + \det B$.

Solution:

False. Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

Then $\det A = \det B = 0$, but $\det(A+B) = \det I_{2\times 2} = 1$.

(e) If V and W are subspaces of $\mathbb{R}^n$ having the same dimension, then $V = W$.

Solution:

False. Consider $V = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ and $W = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. Both V and W are subspaces of $\mathbb{R}^2$ with dimension 1, yet $V \neq W$ since $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is in V but not in W.

(f) If 0 is an eigenvalue of A, then $\det A = 0$.

Solution:

True. If λ is an eigenvalue of A, then $\det(A - \lambda I) = 0$. Thus,

$$\det(A - 0I) = \det(A) = 0.$$

3. (16 pts.) Let A be the matrix given below, where R is the reduced row echelon form of A.

$$A = \begin{bmatrix} -7 & -21 & 4 & -4 & 1 \\ -4 & -12 & -6 & 6 & 1 \\ 1 & 3 & -4 & 4 & 0 \\ 5 & 15 & -4 & 4 & -1 \end{bmatrix} \quad R = \begin{bmatrix} 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find a basis for Row A.

Solution:

The rowspace of A is the same as the rowspace of R, and the nonzero rows of R form a basis for Row R. Thus, the nonzero rows of R also form a basis for Row A:

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(b) Find a basis for Col A^T.

Solution:

The column space of A^T is the same as the row space of A, and so a basis for Col A^T is

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(c) Does the set $\left\{ \begin{bmatrix} -21 \\ -12 \\ 3 \\ 15 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \\ 4 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right\}$ span Col A? (Note that these vectors are the second, fourth, and fifth columns of A, respectively.)

Solution:

We use our method for showing that a subset of vectors is a basis for a subspace. Let $A = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\ \mathbf{a}_4\ \mathbf{a}_5]$, and let $S = \{\mathbf{a}_2, \mathbf{a}_4, \mathbf{a}_5\}$. First note that $\mathbf{a}_2$, $\mathbf{a}_4$, and $\mathbf{a}_5$ are columns of A and hence in Col A.

Second, we must determine if S is linearly independent. Let $R = [\mathbf{r}_1\ \mathbf{r}_2\ \mathbf{r}_3\ \mathbf{r}_4\ \mathbf{r}_5]$. By the Linear Correspondence Property, the set $S' = \{\mathbf{r}_2, \mathbf{r}_4, \mathbf{r}_5\}$ is linearly independent if and only if S is linearly independent. But S' is clearly linearly independent since S' consists of scalar multiples of the standard basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ in $\mathbb{R}^3$. Thus, S is linearly independent.

Third, S must contain exactly k vectors, where k is the dimension of Col A. The dimension of Col A is the rank of A, which we can see from R is 3. S contains 3 vectors, and so by Theorem 4.5, S is a basis for Col A.

4. (20 pts.) Determine if each matrix is diagonalizable (over the reals). If the matrix is diagonalizable, find the matrices P and D such that the matrix is equal to PDP^{-1}. If the matrix is not diagonalizable, explain why not.

(a) $A = \begin{bmatrix} 1 & -1 \\ 4 & 3 \end{bmatrix}$

Solution:

The characteristic polynomial of A is

$$\begin{aligned} \det(A - tI) &= \det \begin{bmatrix} 1-t & -1 \\ 4 & 3-t \end{bmatrix} \\ &= (1-t)(3-t) - 4(-1) \\ &= (3 - 4t + t^2) + 4 \\ &= t^2 - 4t + 7. \end{aligned}$$

From the quadratic formula,

$$t = \frac{4 \pm \sqrt{(-4)^2 - 4 \cdot 7}}{2},$$

which is complex since we are taking the square root of a negative number. Thus, A has no real eigenvalues, and so is not diagonalizable over the reals.

(b) $B = \begin{bmatrix} 7 & 5 \\ -10 & -8 \end{bmatrix}$

Solution:

The characteristic polynomial of B is

$$\begin{aligned} \det(B - tI) &= \det \begin{bmatrix} 7-t & 5 \\ -10 & -8-t \end{bmatrix} \\ &= (7-t)(-8-t) - (-50) \\ &= t^2 + t - 6 \\ &= (t+3)(t-2). \end{aligned}$$

Thus, the eigenvalues of B are 2 and -3, both of multiplicity 1. Since B has n (in this case $n = 2$) distinct eigenvalues, B is diagonalizable.

To form the matrix P, we need to find eigenvectors that correspond to each of the eigenvalues. For the eigenvalue 2, such an eigenvector is a solution to the system $(B - 2I)\mathbf{x} = \mathbf{0}$.

$$B - 2I = \begin{bmatrix} 5 & 5 \\ -10 & -10 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Thus, a basis for the eigenspace is

$$\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}.$$

For the eigenvalue -3, we want to find a solution to the system $(B + 3I)\mathbf{x} = \mathbf{0}$.

$$B + 3I = \begin{bmatrix} 10 & 5 \\ -10 & -5 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix}$$

Thus, a basis for the eigenspace is

$$\left\{ \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} \right\}.$$

We form the matrix P by putting the eigenvectors as the columns of P, and we form the diagonal matrix D by putting the corresponding eigenvalues down the diagonal. Thus,

$$P = \begin{bmatrix} -1 & -1/2 \\ 1 & 1 \end{bmatrix}, \text{ and } \quad D = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}.$$

(c) $C = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$

Solution:

The characteristic polynomial of C is

$$\begin{aligned} \det(C - tI) &= \det \begin{bmatrix} 1-t & 1 \\ -1 & 3-t \end{bmatrix} \\ &= (1-t)(3-t) - 1 \\ &= t^2 - 4t + 4 \\ &= (t-2)^2. \end{aligned}$$

Thus, 2 is an eigenvalue of multiplicity 2.

We now wish to determine if the dimension of the eigenspace corresponding to 2 has the same dimension as the multiplicity. The dimension of the eigenspace is the dimension of Null $(C - 2I)$, which is just nullity $(C - 2I)$. Since we have

$$C - 2I = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix},$$

nullity $(C - 2I) = 1$. But the multiplicity of 2 is 2. Thus, we will not be able to find a basis for $\mathbb{R}^2$ consisting of eigenvectors of C, and so C is not diagonalizable.

5. (10 pts.) Let $A = \begin{bmatrix} 0.6 & 0.1 \\ 0.4 & 0.9 \end{bmatrix}$ be the transition matrix of a Markov chain.

(a) Is the Markov chain associated to A regular? Explain why or why not. What does this tell you about the eigenvalues of A?

Solution:

A Markov chain is regular if it is possible (*i.e.*, with nonzero probability) to eventually get from any state to any other state. A sufficient condition for a Markov chain to be regular is that the transition matrix associated with the chain has no zero entries. Since this is the case here, the Markov chain associated to A is regular.

From a theorem discussed in class and recorded on the formula sheet, 1 is an eigenvalue of the transition matrix of a regular Markov chain.

(b) Find an eigenvector of A that is also a probability vector.

Solution:

From the previous question, we know that 1 is an eigenvalue of A. To find an eigenvector corresponding to 1, we wish to solve $(A - I)\mathbf{x} = \mathbf{0}$.

$$A - I = \begin{bmatrix} -0.4 & 0.1 \\ 0.4 & -0.1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -.25 \\ 0 & 0 \end{bmatrix}$$

Thus, $\left\{ \begin{bmatrix} .25 \\ 1 \end{bmatrix} \right\}$ is a basis for the eigenspace corresponding to the eigenvalue 1.

A probability vector has the property that the sum of its components is 1. To find a probability vector in the eigenspace, we scale any eigenvector by 1 over the sum of its components:

$$\mathbf{p} = \frac{1}{.25 + 1} \begin{bmatrix} .25 \\ 1 \end{bmatrix} = \begin{bmatrix} .2 \\ .8 \end{bmatrix}.$$

6. (20 pts.) Prove the following statements.

(a) Let $\mathcal{S} = \{\mathbf{u}_1, \mathbf{u}_2\}$ be a set of vectors in $\mathbb{R}^n$. Prove that the span of $\mathcal{S}$ is a subspace of $\mathbb{R}^n$.

Solution:
To show that a subset of $\mathbb{R}^n$ is a subspace, we need to verify that the following three properties hold for the subset:

i. $\mathbf{0} \in \text{span } \mathcal{S}$, since $\mathbf{0} = 0\mathbf{u}_1 + 0\mathbf{u}_2$.

ii. span $\mathcal{S}$ is closed under vector addition: Suppose that

$$\begin{aligned} \mathbf{v} &= c_1\mathbf{u}_1 + c_2\mathbf{u}_2 \quad \text{and that} \\ \mathbf{w} &= d_1\mathbf{u}_1 + d_2\mathbf{u}_2. \end{aligned}$$

Then

$$\mathbf{v} + \mathbf{w} = (c_1 + d_1)\mathbf{u}_1 + (c_2 + d_2)\mathbf{u}_2$$

and so $\mathbf{v} + \mathbf{w}$ is in span $\mathcal{S}$.

iii. span $\mathcal{S}$ is closed under scalar multiplication: Suppose that

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2,$$

and that d is any scalar. Then

$$d\mathbf{v} = (dc_1)\mathbf{u}_1 + (dc_2)\mathbf{u}_2$$

and so $d\mathbf{v}$ is in span $\mathcal{S}$.

Since the above properties are satisfied, span $\mathcal{S}$ is a subspace.

(b) If A is diagonalizable, prove that A^T is diagonalizable.

Solution:
Since A is diagonalizable, there exist an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. Applying transpose to both sides, we have that

$$\begin{aligned} A^T &= (PDP^{-1})^T \\ &= (P^{-1})^T D^T P^T \\ &= (P^T)^{-1} D P^T \quad \text{since } D \text{ is diagonal, } D^T = D \\ &= [(P^T)^{-1}] D [(P^T)^{-1}]^{-1} \end{aligned}$$

Since $(P^T)^{-1}$ is invertible, we have written A^T in the form QDQ^{-1} for an invertible matrix Q and a diagonal matrix D. Thus, A^T is diagonalizable.