Math 135:04, Quiz #8 | 10 points | Name ________________________
Mar. 28, 2002 | Show all work!

1. (6 pts.) Let $f(x) = \dfrac{x^3 - x}{x(x-1)(x+2)}$. Find all the horizontal and vertical asymptotes of $f(x)$. If a type of asymptote does not exist, explain why not.

Solution:

The line $y = b$ is a horizontal asymptote if either $\lim_{x\to\infty} f(x) = b$ or $\lim_{x\to-\infty} f(x) = b$.

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{x^3 - x}{x(x-1)(x+2)} \\
&= \lim_{x\to\infty} \frac{x^3 - x}{x^3 + x^2 - 2x}\left(\frac{1/x^3}{1/x^3}\right) \\
&= \lim_{x\to\infty} \frac{1 - 1/x^3}{1 + 1/x - 2/x^2} \\
&= 1.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to-\infty} \frac{x^3 - x}{x(x-1)(x+2)} \\
&= \lim_{x\to-\infty} \frac{1 - 1/x^3}{1 + 1/x - 2/x^2} \\
&= 1.
\end{aligned}$$

Thus, the line $y = 1$ is a horizontal asymptote for $f(x)$.

The line $x = a$ is a vertical asymptote if either $\lim_{x\to a^+} f(x) = \infty$ or $-\infty$, or $\lim_{x\to a^-} f(x) = \infty$ or $-\infty$. For a rational function $f(x) = \dfrac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are both polynomials, then the line $x = a$ is a vertical asymptote if $Q(a) = 0$ but $P(a) \neq 0$. In our case, $P(x) = x^3 - x$, and $Q(x) = x(x-1)(x+2)$. $Q(x) = 0$ when $x = 0, 1,$ or -2. Since $P(-2) = -6 \neq 0$, the line $x = -2$ is a vertical asymptote for $f(x)$. Since $P(0) = P(1) = 0$, the test for rational functions gives no information as to whether $x = 0$ and $x = 1$ are vertical asymptotes. We thus must use the limit definition:

$$\begin{aligned}
\lim_{x\to 0} f(x) &= \lim_{x\to 0} \frac{x^3 - x}{x(x-1)(x+2)} = \lim_{x\to 0} \frac{x^2 - 1}{(x-1)(x+2)} = \frac{1}{2}. \\
\lim_{x\to 1} f(x) &= \lim_{x\to 1} \frac{x^3 - x}{x(x-1)(x+2)} = \lim_{x\to 1} \frac{x(x+1)}{x(x+2)} = \frac{2}{3}.
\end{aligned}$$

Since none of these limits are ∞ or $-\infty$, $x = 0$ and $x = 1$ are *not* vertical asymptotes.

2. (4 pts.) Section 4.4, #15. Let $f(x) = x^2 - 2x - 3$, defined on $[-2, 3]$. Find the absolute maximum value and the absolute minimum, if any, of $f(x)$.

   Solution:

   First, we find the critical points of f that lie in the open interval $(-2, 3)$.

   $$f'(x) = 2x - 2 = 0 \text{ when } x = 1.$$

   Next, we compute f evaluated at each critical point, along with the endpoints of the closed interval. The absolute maximum and minimum correspond to the largest and smallest, respectively, of these values.

   $$\begin{aligned} f(-2) &= 5, \\ f(1) &= -4, \\ f(3) &= 0. \end{aligned}$$

   Thus, the absolute maximum of $f(x)$ on $[-2, 3]$ is 5, and the absolute minimum is -4.