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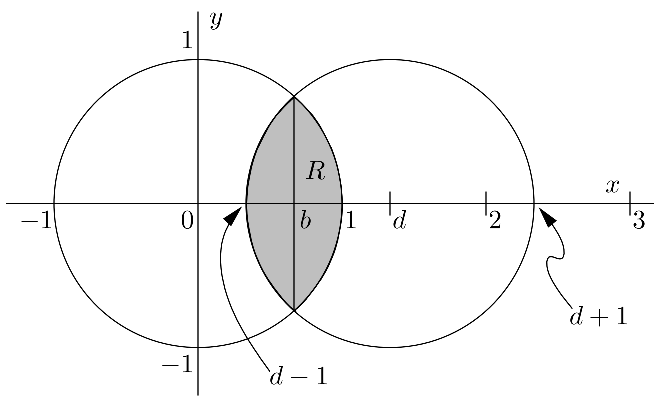
Math 152:18 Calculus II
Workshop #5 Problem 5
October 2, 2002

Problem 5: Find to four place accuracy the number d so that if the center of two circles of radius 1 are at distance d, the area common to the two circles is half of the area of either circle.

Solution: We first sketch the two circles. For simplicity, we center the first circle on the origin, and place the second circle on the x-axis at a distance of d. This means that the second circle is centered at $(d, 0)$.



The region common to both areas is shaded in the figure. We wish to express the area A of that region as a function of d. Note that by symmetry A is 4 times the area of the region labeled R. Now, the equation of the left circle is $x^2 + y^2 = 1$, so the area of R is given by the following integral:

$$\text{Area of } R = \int_b^1 \sqrt{1 - x^2}\, dx.$$

What is b? Since the two circles are the same size, our figure is symmetric about the line $x = b$, and hence b must be midway between the two centers of the circles. Thus, $b = d/2$.

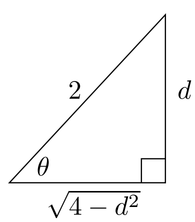
Now we can evaluate the integral using the trigonometric substitution $x = \sin(\theta)$.

$$\begin{aligned}
\text{Area of } R &= \int_{d/2}^{1} \sqrt{1-x^2}\ dx \\
&= \int \sqrt{1-\sin^2(\theta)}\ (\cos(\theta)\ d\theta) \quad \text{(note that } dx = \cos(\theta)\ d\theta) \\
&= \int \cos^2(\theta)\ d\theta \\
&= \int \frac{1}{2}(1+\cos(2\theta))\ d\theta \\
&= \left[\frac{\theta}{2} + \frac{\sin(2\theta)}{4}\right]_{x=d/2}^{x=1} \\
&= \left[\frac{\arcsin(x)}{2} + \frac{\sin(2\arcsin(x))}{4}\right]_{d/2}^{1} \\
&= \left[\frac{\pi}{4} + \frac{\sin(\pi)}{4}\right] - \left[\frac{1}{2}\arcsin(d/2) + \frac{1}{4}\sin(2\arcsin(d/2))\right]. \quad \text{(since } \arcsin(1) = \pi/2)
\end{aligned}$$

We can also simplify a bit further:

$$\begin{aligned}
\sin(2\arcsin(d/2)) &= 2\sin(\arcsin(d/2))\cos(\arcsin(d/2)) \\
&= d\cos(\arcsin(d/2)).
\end{aligned}$$

To find $\cos(\arcsin(d/2))$, we draw a right triangle with $\sin(\theta) = d/2$.



From the picture, we see that $\cos(\arcsin(d/2))$ is $\frac{1}{2}\sqrt{4-d^2}$. Thus,

$$\begin{aligned}
\text{Area of } R &= \frac{\pi}{4} - \frac{1}{2}\arcsin(d/2) - \frac{1}{8}d\sqrt{4-d^2}, \\
\text{Area of the shaded region} &= 4(\text{Area of } R) \\
&= \pi - 2\arcsin(d/2) - \frac{1}{2}d\sqrt{4-d^2}.
\end{aligned}$$

We want to find the value of d such that the area of the shaded region is half the area of either circle. The area of one of the circles is π, since each circle has radius 1, so we have the following equation:

$$\begin{aligned}
\text{Area of the shaded region} &= \frac{1}{2}(\text{Area of one circle}) \\
\pi - 2\arcsin(d/2) - \frac{1}{2}d\sqrt{4-d^2} &= \pi/2 \\
\text{Simplifying,} \quad 2\arcsin(d/2) + \frac{1}{2}d\sqrt{4-d^2} - \pi/2 &= 0.
\end{aligned}$$

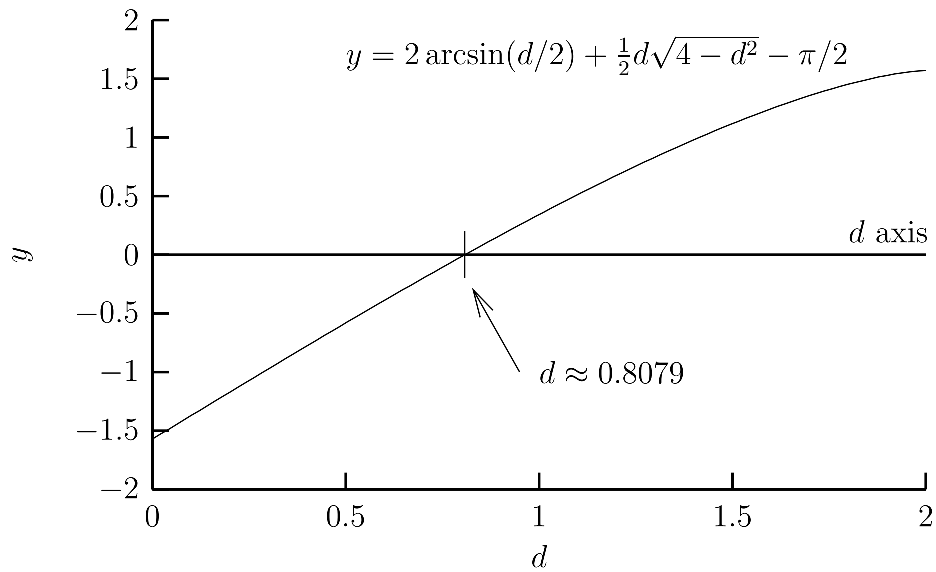
Since we cannot solve the above equation for d directly, we can use a calculator or a computer program to get a numerical approximation for d. I used the command `fsolve` in the computer algebra system Maple which is available in the computer labs and on the Rutgers Eden system.

$$d \approx .8079.$$

Alternatively, we can consider the graph of the function

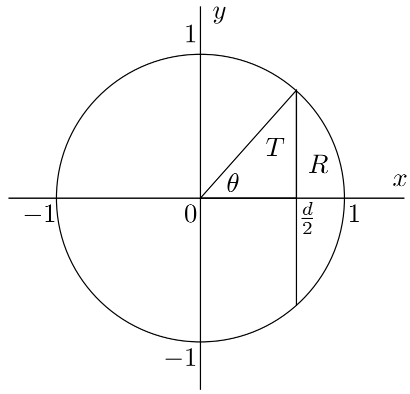
$$f(d) = 2\arcsin(d/2) + \frac{1}{2}d\sqrt{4-d^2} - \pi/2.$$

A reasonable range for d is $[0, 2]$. We can estimate the zero of $f(d)$ either visually, by using an iterative approximation technique such as Newton's method, or again with the aid of a calculator or computer program.



An alternate derivation of the area of R:

We can derive the formula for the area of R without using calculus.



From the figure, we see that

$$\text{Area of } R = (\text{Area of the sector of the circle determined by } \theta) - (\text{Area of the triangle } T).$$

Now,

$$\begin{aligned} \text{Area of the sector of the circle determined by } \theta &= \frac{\theta}{2\pi}(\text{area of the circle}) \\ &= \frac{\theta}{2\pi}(\pi) \\ &= \theta/2. \end{aligned}$$

To relate θ to d, we realize that $\cos(\theta) = d/2$. We can also use the identity $\cos(\theta) = \sin(\pi/2 - \theta)$ to write this as $\sin(\pi/2 - \theta) = d/2$. Solving for θ we get

$$\theta = \pi/2 - \arcsin(d/2).$$

Calculating the area of T, we get

$$\begin{aligned} \text{Area of the triangle } T &= \frac{1}{2}(\text{base} \times \text{height}) \\ &= \frac{1}{2}(d/2)(\sqrt{1-(d/2)^2}) \\ &= \frac{1}{8}d\sqrt{4-d^2}. \end{aligned}$$

Thus, our formula for the area of R is

$$\text{Area of } R = \frac{\pi}{4} - \frac{1}{2}\arcsin(d/2) - \frac{1}{8}d\sqrt{4-d^2}.$$