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Math 152:18 Calculus II
Workshop #3 Problem 8
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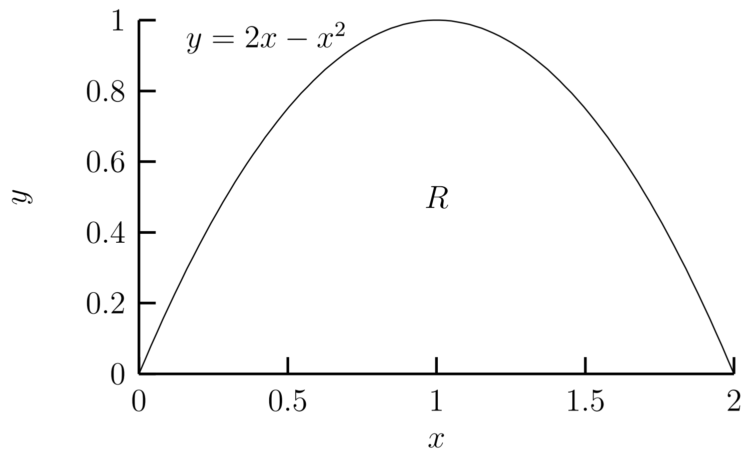
Problem 8: Let R be the first quadrant region bounded by the parabola $y = 2x - x^2$ and the x-axis. Find the constant m so that the line $y = mx$ divides R into two regions of the same area.

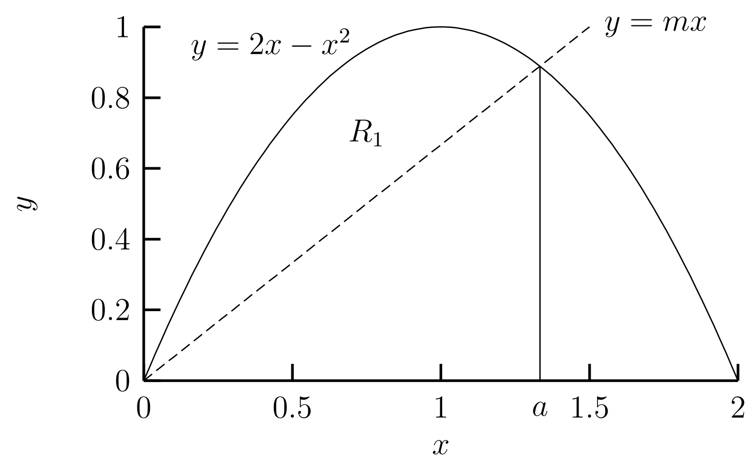
Solution: First, we sketch the region R.



Next, we calculate the area of R. Note that $p = 2x - x^2$ intersects the x-axis at $x = 0$ and $x = 2$. Thus, we can calculate the area by integrating $2x - x^2$ from 0 to 2.

$$\begin{aligned}\text{Area of } R &= \int_0^2 2x - x^2 \, dx \\ &= \left[x^2 - \frac{x^3}{3}\right]_0^2 \\ &= 4 - \frac{8}{3} = \frac{4}{3}.\end{aligned}$$

Now we wish to find the value of m such that the line $y = mx$ divides R into two regions of equal area.



One way to proceed is to find the area of the region R_1 bounded above by the parabola and below by the line, and setting this area to be one half of the total area of R. The area of R_1 can be found using the integral

$$\int_0^a (2x - x^2) - mx \; dx,$$

where a is the x coordinate of the point of intersection of the parabola and the line. Let's find a. We can find all the points of intersection of the two curves by setting them equal and solving for x.

$$\begin{aligned} mx &= 2x - x^2 \\ x^2 + (m-2)x &= 0 \\ x &= 0 \text{ or } 2 - m. \end{aligned}$$

Since we already know that one of the intersections occurs at the origin, we can discard 0. Thus, $a = 2 - m$.

Now we can find the area of R_1 in terms of the unknown a. (We could also solve for the unknown m, but it turns out the calculations are easier if we solve for a.)

$$\begin{aligned} \text{Area of } R_1 &= \frac{1}{2}(\text{Area of } R) \\ \int_0^a (2x - x^2) - mx \; dx &= \frac{1}{2}\left(\frac{4}{3}\right) \\ \int_0^a -x^2 + ax \; dx &= \frac{2}{3} \qquad \text{since } a = 2 - m \\ \left[\frac{-x^3}{3} + \frac{ax^2}{2}\right]_0^a &= \frac{2}{3} \\ \frac{a^3}{6} &= \frac{2}{3} \\ a &= 4^{1/3} \end{aligned}$$

Now that we know a, we also know m: $m = 2 - 4^{1/3}$. Thus, the line $y = (2 - 4^{1/3})x$ divides the region R into two regions of equal area.