Quiz #2
Math 152:18 Calculus II
October 2, 2002

1. (5 pts.) Integrate $\int \frac{x}{x^4 - a^4}\, dx$. (Hint: let $u = x^2$.)

   Solution:

$$
\begin{aligned}
\int \frac{x}{x^4 - a^4}\, dx &= \int \frac{1}{u^2 - a^4} \frac{du}{2} \quad \text{from substituting } u = x^2,\ du = 2\ dx \\
&= \frac{1}{2} \int \frac{\frac{-1}{2a^2}}{u + a^2} + \frac{\frac{1}{2a^2}}{u - a^2}\, du \quad \text{using partial fractions} \\
&= \frac{-1}{4a^2} \ln|u + a^2| + \frac{1}{4a^2} \ln|u - a^2| + C \\
&= \frac{-1}{4a^2} \ln|x^2 + a^2| + \frac{1}{4a^2} \ln|x^2 - a^2| + C
\end{aligned}
$$

2. (5 pts.) Determine if the integral $\int_1^\infty \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\, dx$ is convergent or divergent. Evaluate if convergent.

   Solution 1:

   We make the substitution $u = 1 + \sqrt{x}$, $du = \frac{dx}{2\sqrt{x}}$:

$$
\begin{aligned}
\int_1^\infty \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\, dx &= \lim_{a\to\infty} \int_1^a \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\, dx \\
&= \lim_{a\to\infty} \int_0^{\sqrt{a-1}} 2\sqrt{u}\, du \\
&= \lim_{a\to\infty} \left[\frac{4}{3} u^{3/2}\right]_1^{\sqrt{a}-1} \\
&= \lim_{a\to\infty} \left[\frac{4}{3}(\sqrt{a} - 1)^{3/2} - \frac{4}{3}\right] \\
&= \infty.
\end{aligned}
$$

   Thus, the integral diverges.

   Solution 2:

   We first note that

$$
\begin{aligned}
1 + \sqrt{x} &\geq \sqrt{x} \quad \text{for all } x \geq 1 \\
\sqrt{1+\sqrt{x}} &\geq \sqrt{\sqrt{x}} = x^{1/4} \quad \text{since } \sqrt{x} \text{ is an increasing function} \\
\frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}} &\geq \frac{x^{1/4}}{\sqrt{x}} = x^{-1/4} \quad \text{since } \sqrt{x} \geq 0 \text{ for all } x \geq 1.
\end{aligned}
$$

Since

$$\begin{aligned}
\int_1^\infty x^{-1/4}\ dx &= \lim_{a\to\infty} \int_1^a x^{-1/4}\ dx \\
&= \lim_{a\to\infty} \left[\frac{4}{3}a^{3/4} - \frac{4}{3}\right] \\
&= \infty,
\end{aligned}$$

then by the Comparison Theorem, $\displaystyle\int_1^\infty \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\ dx$ also diverges.