

k -Ordered Hamiltonicity of Iterated Line Graphs

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Abstract

A graph G of order n is k -ordered hamiltonian, $2 \leq k \leq n$, if for every sequence v_1, v_2, \dots, v_k of k distinct vertices of G , there exists a hamiltonian cycle that encounters v_1, v_2, \dots, v_k in this order. In this paper, we generalize two well-known theorems of Chartrand on hamiltonicity of iterated line graphs to k -ordered hamiltonicity. We prove that if $L^n(G)$ is k -ordered hamiltonian and n is sufficiently large, then $L^{n+1}(G)$ is $(k+1)$ -ordered hamiltonian. Furthermore, for any connected graph G , which is not a path, cycle, or the claw $K_{1,3}$, there exists an integer N' such that $L^{N'+(k-3)}(G)$ is k -ordered hamiltonian for $k \geq 3$.

1 Introduction

Hamiltonicity is a well-studied property of graphs. Ng and Schultz [9] generalized this notion by defining a graph G of order n to be **k -ordered hamiltonian**, $2 \leq k \leq n$, if for every sequence v_1, v_2, \dots, v_k of k distinct vertices of G , there exists a hamiltonian cycle that encounters v_1, v_2, \dots, v_k in this order. Note that any graph that is hamiltonian is 3-ordered hamiltonian, since both the starting vertex and the orientation of a hamiltonian cycle can be chosen. Also, any graph that is k -ordered hamiltonian is j -ordered hamiltonian for $2 \leq j \leq k$.

The line graph operator is also well-studied in graph theory. Given a graph G , we form the line graph $L(G)$ as follows: the set of vertices of $L(G)$ consists of the set of edges of G , and two vertices of $L(G)$ are adjacent whenever the corresponding edges of G are incident on a common vertex. We write $L^2(G)$ to denote $L(L(G))$, and in general $L^n(G) = L(L^{n-1}(G))$ for $n \geq 1$ (where $L^0(G) = G$).

Chartrand proved two major results regarding the hamiltonicity of line graphs:

Theorem 1 ([2]). *If a graph G is hamiltonian, then $L(G)$ is hamiltonian. Thus, $L^n(G)$ is hamiltonian for all $n \geq 1$.*

Theorem 2 ([3]). *If a graph G is connected and not a path, then there exists an integer N such that $L^N(G)$ is hamiltonian. By Theorem 1, $L^n(G)$ is thus hamiltonian for all $n \geq N$.*

In this paper we generalize these results to k -ordered hamiltonicity. We show that if $L^n(G)$ is k -ordered hamiltonian and n is sufficiently large, then $L^{n+1}(G)$ is $(k+1)$ -ordered hamiltonian. Furthermore, for any connected graph G which is not a path, cycle, or the claw $K_{1,3}$, there exists an integer N' such that $L^{N'+(k-3)}(G)$ is k -ordered hamiltonian for $k \geq 3$.

In our work we denote the minimum degree of a graph G by $\delta(G)$. If two edges e and f in G are incident on a common vertex, then we say that e and f are **adjacent** in G . Note that e and f are then also adjacent, as vertices, in $L(G)$. Throughout this paper, we will consider only simple connected graphs that are not paths, cycles, or the claw $K_{1,3}$.

2 Preliminaries

We begin with some lemmas about finding special sets of triangles in the iterated line graph.

Definition 3. An edge e in $L^{n+1}(G)$ corresponds to two adjacent edges in $L^n(G)$ incident on a common vertex v . We define $S(e)$ to be the clique in $L^{n+1}(G)$ corresponding to all of the edges in $L^n(G)$ incident to v .

Lemma 4. *For every edge e in $L^{n+1}(G)$, the clique $S(e)$ containing e has at least $\delta(L^n(G))$ vertices.*

Proof. The number of vertices in $S(e)$ is equal to the degree of v in $L^n(G)$; this is at least $\delta(L^n(G))$. □

Each vertex in $L^{n+1}(G)$ corresponds to an edge in $L^n(G)$. For a hamiltonian cycle in $L^{n+1}(G)$ that encounters k vertices in a specific order, there are k edges in $L^n(G)$ that correspond to these k vertices. We call these edges **ordered edges** of $L^n(G)$ and the ordering of the k chosen vertices in $L^{n+1}(G)$ creates an ordering for the ordered edges in $L^n(G)$.

For any edge e and a triangle T which contains e , the **opposite vertex** of e in T is the vertex in T that is not one of the endvertices of e .

Lemma 5. *If e_1, \dots, e_k in $L^n(G)$ are ordered edges and $\delta(L^{n-1}(G)) \geq 3k+1$, we can find a set $\{T_1, T_2, \dots, T_{2k}\}$ of distinct triangles in $L^n(G)$ such that*

1. *For every ordered edge e_i , there are exactly two triangles, called T_{2i-1} and T_{2i} , that contain the edge e_i .*
2. *Each triangle contains exactly one of the ordered edges e_i .*
3. *Two triangles share either no edges or exactly one edge which is one of the k ordered edges e_1, \dots, e_k .*
4. *The set $\{v \in V(L^n(G)) \mid v \text{ is opposite } e_i \text{ in } T_{2i-1} \text{ or } T_{2i}\}$ of vertices in each triangle opposite the ordered edges is a set of $2k$ distinct vertices.*

Proof. We show that if $\delta(L^{n-1}(G)) \geq 3k + 1$, there are a sufficient number of triangles so that two triangles T_{2i-1} and T_{2i} can be assigned to each ordered edge e_i in such a way that the conditions of the lemma are satisfied.

Let e be a given edge in $L^n(G)$. By Lemma 4, $S(e)$ is a clique with at least $\delta(L^{n-1}(G))$ vertices, and hence there are at least $\delta(L^{n-1}(G)) - 2$ triangles in $S(e)$ that contain the edge e .

We begin constructing the set of triangles with edge e_1 . Assign to e_1 any two triangles in $S(e_1)$ that do not contain e_2, \dots, e_k . This can be done since e_1 and e_i , for $2 \leq i \leq k$, can be in at most one triangle together, and hence at most $k - 1$ triangles in $S(e_1)$ cannot be assigned to e_1 . Because there are at least $\delta(L^{n-1}(G)) - 2 \geq 3k - 1$ triangles in $S(e_1)$ that contain e_1 , there are at least $2k$ triangles in $S(e_1)$ that contain e_1 and that avoid e_2, \dots, e_k . Any two such triangles can be assigned to e_1 as T_1 and T_2 .

We now proceed inductively. To choose two triangles T_{2i-1} and T_{2i} for $e_i = v_{i'}v_{i''}$, $2 \leq i \leq k$, we must avoid triangles assigned to previous ordered edges, as well as avoiding all of the ordered edges themselves. We consider two cases based on whether a previous ordered edge is adjacent to e_i or not.

First, suppose that $v_{i'}v_{i''}$ is an ordered edge where e_i and $v_{i'}v_{i''}$ are incident on the common vertex $v_{i'}$. If $v_{i''}$ and $v_{i''}$ are adjacent, then the triangle $v_{i'}v_{i''}v_{i''}$ contains e_i but cannot be assigned to e_i because it also contains the ordered edge $v_{i'}v_{i''}$.

Suppose two triangles $v_{i'}v_{i''}v_{m'}$ and $v_{i'}v_{i''}v_{m''}$ have been assigned to the ordered edge $v_{i'}v_{i''}$. Then the two triangles $v_{i'}v_{i''}v_{m'}$ and $v_{i'}v_{i''}v_{m''}$, if they exist in the graph, cannot be assigned to e_i since the triangles share the edges $v_{i'}v_{m'}$ and $v_{i'}v_{m''}$ with the triangles assigned to $v_{i'}v_{i''}$. Note that if a triangle assigned to e_i and a triangle assigned to $v_{i'}v_{i''}$ violate condition 4, then they also violate condition 3. Thus, every other ordered edge may eliminate at most three triangles from the set of triangles in $S(e_i)$ that can be assigned to e_i .

Second, suppose that v_jv_ℓ is an ordered edge where v_jv_ℓ is not adjacent to e_i . Suppose two triangles $v_jv_\ell v_{m'}$ and $v_jv_\ell v_{m''}$ have been assigned to the ordered edge v_jv_ℓ . If $m' = i'$, then the triangle $v_{i'}v_{i''}v_j$ cannot be assigned to e_i because the edge $v_{i'}v_j$ is shared. Similarly, the triangle $v_{i'}v_{i''}v_\ell$ cannot be assigned to e_i , and, if $m' = i''$, then the same two triangles $v_{i'}v_{i''}v_j$ and $v_{i'}v_{i''}v_\ell$ cannot be assigned to e_i . If $m' \notin \{i', i''\}$, then the triangle $v_{i'}v_{i''}v_{m'}$ cannot be assigned to e_i since the shared vertex $v_{m'}$ would then violate condition 4. Similar cases hold for the triangle $v_jv_\ell v_{m''}$. Thus, two triangles are eliminated from the set of triangles in $S(e_i)$ that can be assigned to e_i if $|\{m', m''\} \cap \{i', i''\}| = 0$, three triangles if the intersection size is 1, and two triangles if the intersection size is 2.

In both cases, for any ordered edge e_i every other ordered edge reduces the number of triangles that can be assigned to e_i by at most three. Since

$$\begin{aligned} \delta(L^{n-1}(G)) - 2 &\geq 3k - 1 && \text{(by hypothesis)} \\ &\geq 3(k - 1) + 2, \end{aligned}$$

there are enough triangles in $S(e_i)$ to assign two triangles to e_i that satisfy the properties in the statement, for each $2 \leq i \leq k$. Since there must be at least two allowed triangles that can be assigned to each ordered edge, the number of triangles needed for each ordered edge is at most $3(k - 1) + 2 = 3k - 1$. The assigned triangles then form the set $\{T_1, T_2, \dots, T_{2k}\}$. \square

Lemma 6. *Let e_1, \dots, e_k be k ordered edges in a graph $L^n(G)$. By Lemma 5, each of these edges e_i has two distinct triangles T_{2i-1} and T_{2i} assigned to it that satisfy the conditions of that lemma. Assume there is a hamiltonian cycle in $L^n(G)$ and that*

$$\left\lfloor \frac{\delta(L^{n-1}(G)) - 3}{4} \right\rfloor \geq 3k + 1.$$

Then we can find a set $\{U_1, U_2, \dots\}$ of distinct triangles in $L^n(G)$ such that

1. For each edge h_j of the hamiltonian cycle, there is exactly one triangle, called U_j , that contains the edge h_j .
2. Each triangle contains exactly one of the edges in the hamiltonian cycle.
3. Any two of these triangles are edge-disjoint.
4. Let M be the set of k ordered edges e_1, \dots, e_k along with the edges in the assigned triangles T_1, \dots, T_{2k} . A triangle U_j can contain at most one edge from M , and then only if that edge is an ordered edge in the hamiltonian cycle.

Proof. For each edge h_j in the hamiltonian cycle, we want to choose a triangle U_j within $S(h_j)$ that contains h_j and does not contain any of the ordered edges (except h_j if h_j is an ordered edge); edges in triangles T_{2i-1} or T_{2i} , $1 \leq i \leq k$; edges contained in any triangle U_l assigned to h_l , $l \neq j$; or h_l , $l \neq j$. Consider two edges e and f in $L^n(G)$. If f is an edge in $S(e)$, then $S(e) = S(f)$. However, if f is not in $S(e)$, then $S(e)$ and $S(f)$ are edge-disjoint. Therefore the triangles within $S(e)$ assigned to e are edge-disjoint from the triangles in $S(f)$ assigned to f if f is not in $S(e)$.

Each clique $S(e)$ with q vertices contains at most q edges that are part of the hamiltonian cycle. Label the vertices in $S(e)$ with v_1, \dots, v_q in the order they appear in the hamiltonian cycle. We consider all subscripts of vertices inside $S(e)$ modulo q . Only edges of the form $v_t v_{t+1}$ can appear in the hamiltonian cycle. Note, however, that it may be the case that not all of these edges actually appear in the hamiltonian cycle. Let \mathcal{U}_t be the set of triangles

$$\mathcal{U}_t = \{v_t v_{t+1} v_{t+2m+1} : 1 \leq m \leq \lfloor (q-3)/4 \rfloor\}.$$

Observe that $\mathcal{U}_t \cap \mathcal{U}_{t'} = \emptyset$ for $t \neq t'$, since in any of the constructed triangles there is a unique edge that has consecutively indexed endpoints. By construction, no two triangles in \mathcal{U}_t for a given t share an edge except $v_t v_{t+1}$. Suppose that $v_t v_{t+1} v_{t+2m+1}$ is a triangle in \mathcal{U}_t and that $v_{t'} v_{t'+1} v_{t'+2m'+1}$ is a triangle in $\mathcal{U}_{t'}$, for $t \neq t'$. The unique edges $v_t v_{t+1}$ and $v_{t'} v_{t'+1}$ that have consecutively indexed endpoints are distinct since $t \neq t'$ and $q > 2$. The two triangles share an edge then only if $v_t v_{t+2m+1} = v_{t'} v_{t'+2m'+1}$, $v_t v_{t+2m+1} = v_{t'+1} v_{t'+2m'+1}$, $v_{t+1} v_{t+2m+1} = v_{t'} v_{t'+2m'+1}$, or $v_{t+1} v_{t+2m+1} = v_{t'+1} v_{t'+2m'+1}$. Suppose that $v_t v_{t+2m+1} = v_{t'} v_{t'+2m'+1}$. Then $t = t' + 2m' + 1$ and $t' = t + 2m + 1$ (since $t \neq t'$), which implies that $2(m + m') + 2 = 0$ (considered modulo q). However, this contradicts the facts that $1 \leq m \leq \lfloor (q-3)/4 \rfloor$ and $1 \leq m' \leq \lfloor (q-3)/4 \rfloor$. The other cases follow similarly. Thus, a triangle in \mathcal{U}_t is edge-disjoint from every triangle in $\mathcal{U}_{t'}$ for $t \neq t'$.

We thus have a set \mathcal{U}_t of triangles for each edge $v_t v_{t+1}$ where each triangle contains at most one edge from the hamiltonian cycle and where any two triangles from different sets

are edge-disjoint. We now show that if $\lfloor (\delta(L^{n-1}(G)) - 3)/4 \rfloor \geq 3k + 1$, we can choose one triangle for each h_i from the associated set \mathcal{U}_t of triangles that satisfies conditions 1 and 2 (trivially) and condition 4 of the lemma.

To ensure that the triangle chosen for h_i avoids ordered edges and their assigned triangles, we consider three cases. First, suppose that h_i is an ordered edge $v_t v_{t+1}$ that is part of the hamiltonian cycle. In this case, the triangles $v_t v_{t+1} v_j$ and $v_t v_{t+1} v_k$ assigned to the ordered edge eliminate two triangles from \mathcal{U}_t that could have been assigned to $v_t v_{t+1}$ as an edge in the hamiltonian cycle.

Second, suppose that an ordered edge $v_t v_i$ is adjacent to the edge $h_i = v_t v_{t+1}$ on the hamiltonian cycle, and that the ordered edge has the two triangles $v_t v_i v_j$ and $v_t v_i v_k$ assigned to it. If v_j is adjacent to v_{t+1} , then $v_t v_{t+1} v_j$ cannot be assigned to $v_t v_{t+1}$. The same situation results if v_k is adjacent to v_{t+1} . Additionally, the triangle $v_t v_{t+1} v_i$ cannot be assigned to $v_t v_{t+1}$. Thus, each ordered edge incident to h_i eliminates at most three triangles from \mathcal{U}_t that could have been assigned to $v_t v_{t+1}$ as an edge in the hamiltonian cycle.

Third, suppose that the ordered edge $v_i v_j$ is not adjacent to the edge $h_i = v_t v_{t+1}$ on the hamiltonian cycle. If $v_i v_j$ is assigned either triangle $v_i v_j v_t$ or $v_i v_j v_{t+1}$, then neither triangle $v_t v_{t+1} v_i$ nor $v_t v_{t+1} v_j$ can be assigned to $v_t v_{t+1}$. Thus, each ordered edge not incident to h_i eliminates two triangles from \mathcal{U}_t that could have been assigned to $v_t v_{t+1}$ as an edge in the hamiltonian cycle.

For each edge $h_j = v_t v_{t+1}$, there are at most $3k$ triangles in \mathcal{U}_t that cannot be chosen as U_j . Thus, if $|\mathcal{U}_t| \geq 3k + 1$, there is one triangle in \mathcal{U}_t that satisfies the conditions of the lemma. Since q corresponds to the number of vertices in $S(e)$, which is the degree of e as a vertex in $L^{n-1}(G)$, and since

$$\left\lfloor \frac{\delta(L^{n-1}(G)) - 3}{4} \right\rfloor \geq 3k + 1$$

by hypothesis, $|\mathcal{U}_t| = \lfloor (q - 3)/4 \rfloor \geq 3k + 1$. Hence we can choose a set $\{U_1, U_2, \dots\}$ of triangles that satisfies the conditions of the lemma. \square

3 Main Result

We now proceed to our main result. In Theorem 7 we show that for sufficiently large n , $L^{n+1}(G)$ is $(k + 1)$ -ordered hamiltonian if $L^n(G)$ is k -ordered hamiltonian.

To prove this statement we will find a complete cyclic list of adjacent edges in $L^n(G)$ which contains the given set of k ordered edges in the specified order. A **closed trail** is a sequence of distinct edges $x_1 x_2, x_3 x_4, \dots, x_{\ell-1} x_\ell$ in $L^n(G)$ where $x_{2j} = x_{2j+1}$ for $1 \leq j \leq \ell/2 - 1$ and $x_\ell = x_1$. A **closed list of adjacent edges** is a sequence of distinct edges in $L^n(G)$ where consecutive edges are adjacent and the last edge is adjacent to the first edge. A closed list of adjacent edges in $L^n(G)$ corresponds to a cycle in $L^{n+1}(G)$. Note that every closed trail is a closed list of adjacent edges, but not conversely: for instance, the claw $K_{1,3}$ has a closed list of three adjacent edges, but no closed trail.

A **complete closed list of adjacent edges** is a closed list of adjacent edges where every edge in $L^n(G)$ appears in the sequence. A complete closed list of adjacent edges corresponds to a hamiltonian cycle in $L^{n+1}(G)$, and our constructed complete closed list of adjacent edges

will correspond to a hamiltonian cycle in $L^{n+1}(G)$ that contains the corresponding k vertices in the correct order. The use of a complete closed list of adjacent edges to construct a hamiltonian cycle in $L^{n+1}(G)$ was used by Chartrand in [2] and [3] and is similar to methods used by Harary and Nash-Williams in [4].

One possible approach to proving Theorem 7 begins by assigning to each of the first k ordered edges one of its endvertices, and then using the k -ordered hamiltonicity of $L^n(G)$. This method does not work without modification since many ordered edges may share the same endvertices. However, this can be resolved by assigning endvertices to as many ordered edges as possible and then assigning to the remaining edges the opposite vertex in triangles containing that edge. We will use this last idea as the core of our proof.

Theorem 7. *If $L^n(G)$ is k -ordered hamiltonian and*

$$\left\lfloor \frac{\delta(L^{n-1}(G)) - 3}{4} \right\rfloor \geq 3k + 4$$

then $L^{n+1}(G)$ is $(k + 1)$ -ordered hamiltonian.

We first outline the proof. Let e_1, \dots, e_k, e_{k+1} be $k + 1$ ordered edges of $L^n(G)$ that correspond to the $k + 1$ ordered vertices of $L^{n+1}(G)$. Begin by assigning to each of the first k ordered edges e_i a vertex v_i . The ordering of edges provides an ordering of the vertices assigned to these edges. Since $L^n(G)$ is k -ordered hamiltonian, there is a hamiltonian cycle H that passes through the k vertices v_1, \dots, v_k in the desired order; we use the triangles of Lemma 5 to modify the cycle so that it contains the k ordered edges in the desired order. We then use the triangles of Lemma 6 to insert the ordered edge e_{k+1} . The resulting closed list R' of adjacent edges can then be expanded into a complete closed list of adjacent edges that corresponds to a hamiltonian cycle in $L^{n+1}(G)$ that encounters the $k + 1$ ordered vertices in the desired order.

Proof of Theorem 7. Given $k + 1$ ordered vertices in $L^{n+1}(G)$, let e_1, e_2, \dots, e_{k+1} be the corresponding ordered edges in $L^{n+1}(G)$. By applying Lemma 5 to the first k edges e_1, \dots, e_k , there exists a set $\{T_1, \dots, T_{2k}\}$ of triangles that satisfy the conditions of that lemma.

We wish to find an appropriate assignment of vertices to edges so that we may use the k -ordered hamiltonicity of $L^n(G)$. For $1 \leq i \leq k$, we assign to each ordered edge e_i the vertex v_i that is the opposite vertex of e_i in T_{2i-1} . Note that if $i \neq j$, then $v_i \neq v_j$ by condition 4 of Lemma 5. Also note that the second triangle assigned to each edge has the property that it does not contain any of the other ordered edges or any of the edges in other assigned triangles.

The order of the first k edges produces an order for the vertices v_1, \dots, v_k assigned to them. Since $L^n(G)$ is k -ordered hamiltonian, there exists a hamiltonian cycle in $L^n(G)$ that encounters these k vertices in that order. Let the edges of H be h_1, \dots, h_p . By Lemma 6, for each h_j there exists a triangle U_j satisfying the conditions of that lemma.

We now modify the hamiltonian cycle H to create a closed trail R that contains the ordered edges e_1, \dots, e_k in the order specified. We insert the ordered edge e_i into R near v_i , preserving the order in which R encounters the vertices v_1, \dots, v_k and hence the order in which R traverses the ordered edges e_1, \dots, e_k . For each edge e_i , $1 \leq i \leq k$, let $F(e_i)$ denote

the set of edges of T_{2i-1} and T_{2i} . By Lemma 5, $F(e_i)$ and $F(e_j)$ are edge-disjoint if $i \neq j$, and the vertices opposite the ordered edges in $\{T_1, \dots, T_{2k}\}$ are distinct. There are thirteen ways (up to the symmetry of switching the two endvertices of e_i) in which the hamiltonian cycle can pass through the subgraph induced by $F(e_i)$, as shown in Figure 1. Each of the cases in Figure 1 shows how H reaches the four vertices in $F(e_i)$. Note that e_i is the horizontal thin edge in the middle of the drawing, and that v_i is the opposite vertex of the triangle above e_i .

Form R from H by replacing the edges of H in $F(e_i)$ with the dashed edges indicated in Figure 1. For instance, if the edges $z_1x_1, x_1v_i, v_ix_2, x_2w, wz_2$ appear in H , shown as bold edges in case 1 of Figure 1, then we form R by replacing those edges with the dashed edges $z_1x_1, x_1v_i, v_ix_2, x_2x_1, x_1w, wz_2$. Note that the edge v_iw may be present in the graph, but is considered an edge outside of $F(e_i)$. For instance, if the edges $z_1x_1, x_1v_i, v_iw, wz_2$ appear in H , this falls under case 3.

The replacement of edges is done in $F(e_i)$ for all i . Since $F(e_i)$ and $F(e_j)$ are edge-disjoint if $i \neq j$, these changes can be made independently. Since the $F(e_i)$ are disjoint, e_i is inserted into R such that no other ordered edge e_j or vertex v_j , $j \neq i$, is encountered between e_i and v_i . Thus, the closed trail R contains the k ordered edges in the desired order.

Let a_{2j-1} and a_{2j} be the two edges in U_j that are not on the hamiltonian cycle H (*i.e.*, the edges in U_j that are not h_j), labelled such that a_{2j} is adjacent to $a_{2(j+1)-1}$ (subscripts are considered modulo $2p$). Let Q be the closed trail $a_1, a_2, \dots, a_{2p-1}, a_{2p}$. Note that Q does not contain any of the first k ordered edges e_1, \dots, e_k , edges of the hamiltonian cycle H , or edges from the triangles T_1, \dots, T_{2k} assigned to ordered edges. Specifically, Q is edge-disjoint from R . Since Q encounters every vertex in the hamiltonian cycle H , Q encounters every vertex in $L^n(G)$.

If e_{k+1} is contained in the hamiltonian cycle, by Lemma 6 it has been assigned two edges a_m and a_{m+1} . In R replace e_{k+1} with a_m and a_{m+1} , and in Q replace a_m and a_{m+1} with e_{k+1} . If e_{k+1} is not in H , insert e_{k+1} into Q such that e_{k+1} is adjacent to the edge immediately before and after it in Q (this can be done since Q encounters every vertex in $L^n(G)$). Note that in both cases Q and R are still edge-disjoint and encounter every vertex in $L^n(G)$, and that Q contains e_{k+1} .

Let x be the endvertex of e_k encountered by R after e_k , and let a' be an edge of Q that has x as an endvertex. We form the closed list R' of adjacent edges by inserting the edges of Q starting with a' into R after e_k . Since Q and R are both closed lists of adjacent edges, R' is a closed list of adjacent edges that contains the ordered edges e_1, \dots, e_{k+1} in the desired order. R' also encounters every vertex in $L^n(G)$.

We now expand R' to a complete closed list of adjacent edges R'' . For an edge wx_2 not in R' , let x_1x_2 and x_2x_3 be two consecutive edges in R' (x_1x_2 and x_2x_3 may be the last and first edges in R'). To form R'' we insert wx_2 between x_1x_2 and x_2x_3 . Note that wx_2 is adjacent to both x_1x_2 and x_2x_3 . Thus, R'' is a complete closed list of adjacent edges where the ordered edges e_1, \dots, e_{k+1} appear in the specified order, and hence corresponds to a hamiltonian cycle in $L^{n+1}(G)$ that contains the assigned vertices v_1, \dots, v_{k+1} in the desired order. \square

For a discussion of the following two lemmas about the minimum degree of the iterated line graph see [10] and [7].

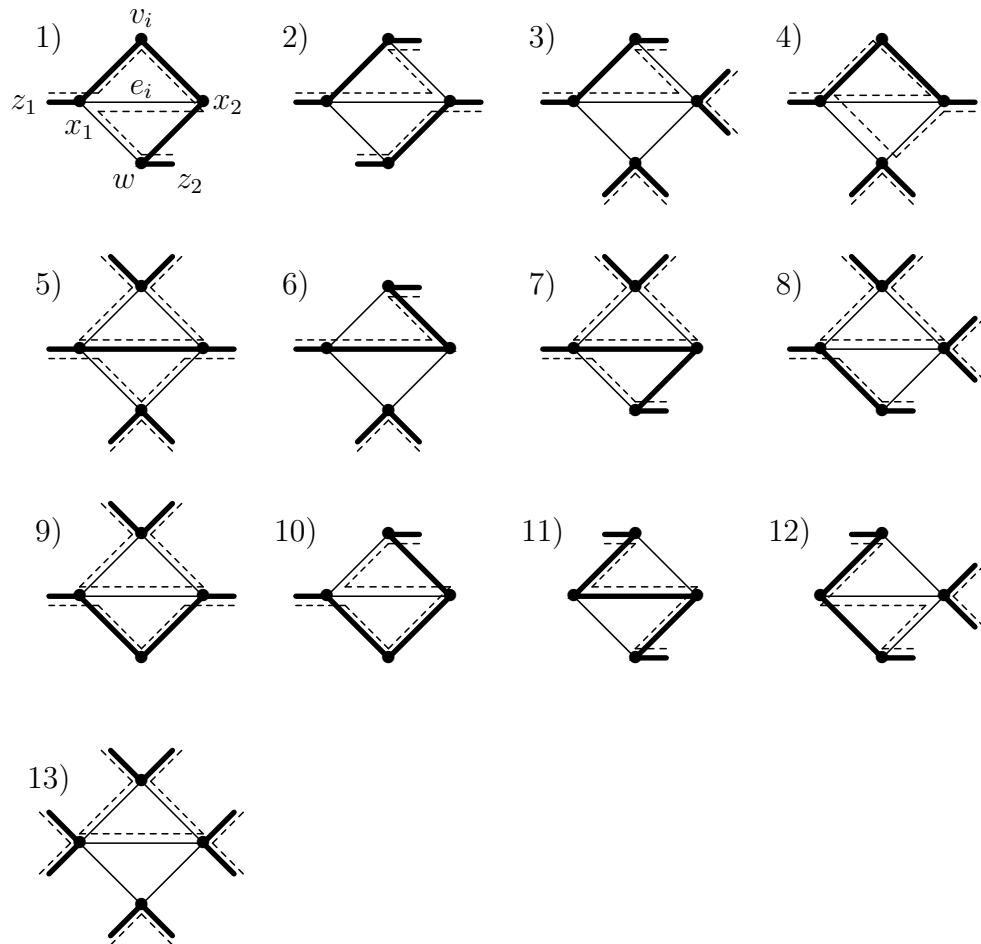


Figure 1: The different ways in which the hamiltonian cycle may pass through $F(e_i)$. The thin solid edges are the edges of $F(e_i)$, the horizontal edge in the middle of each figure is e_i , and the opposite vertex of the triangle above e_i is v_i . The bold lines denote the original edges of the hamiltonian cycle, and the dashed lines indicate the edges used to form the closed list R of adjacent edges.

Lemma 8. *Given a graph G that is not a path, cycle, or the claw $K_{1,3}$, and an integer q , there exists an integer N^* such that $\delta(L^n(G)) \geq q$ for $n \geq N^*$.*

Lemma 9. *If G is not a path, cycle, or the claw $K_{1,3}$, then $\delta(L(G)) \geq 2\delta(G) - 2$. Thus, $\delta(L^n(G)) \geq 2^n(\delta(G) - 2) + 2$.*

Theorem 10. *For any connected graph G , which is not a path, cycle, or the claw $K_{1,3}$, there exists an integer N' such that $L^{N'+(k-3)}(G)$ is k -ordered hamiltonian for $k \geq 3$.*

Proof. By Theorem 2 there exists an integer N large enough such that $L^n(G)$ is hamiltonian (and hence 3-ordered hamiltonian) for $n \geq N$. By Lemma 8, there exists an integer N^* large enough such that $\delta(L^{N^*}(G)) \geq 112$. We choose N' to be the maximum of N and N^* (in fact, the proof of Theorem 2 shows that N^* is always larger than N).

We proceed to verify the conclusion by induction on k . By Theorem 2, $L^{N'}(G)$ is 3-ordered hamiltonian. Thus we assume that $L^{N'+(k-3)}(G)$ is k -ordered hamiltonian for some $k \geq 3$. Note that

$$\begin{aligned} \delta(L^{N'+(k-3)-1}(G)) &\geq 2^{k-4} \left(\delta(L^{N'}(G)) - 2 \right) + 2, && \text{by Lemma 9,} \\ &\geq 112 \cdot 2^{k-4} + 2, \\ &\geq 12k + 22, && \text{since } k \geq 3. \end{aligned}$$

Hence,

$$\left\lfloor \frac{\delta(L^{N'+(k-3)-1}(G)) - 3}{4} \right\rfloor \geq \frac{\delta(L^{N'+(k-3)-1}(G)) - 6}{4} \geq 3k + 4,$$

and by applying Theorem 7, $L^{N'+((k+1)-3)}(G)$ is $(k+1)$ -ordered hamiltonian. \square

Corollary 11. *For any connected graph G , which is not a path, cycle, or the claw $K_{1,3}$, and any k , there exists an integer \hat{N} such that $L^n(G)$ is k -ordered hamiltonian for $n \geq \hat{N}$.*

Proof. Choose \hat{N} to be $N' + (k - 3)$. Then by Theorem 10, $L^n(G)$ is $(n - N' + 3)$ -ordered hamiltonian for $n \geq \hat{N}$, which implies that $L^n(G)$ is k -ordered hamiltonian for $n \geq \hat{N}$. \square

4 Open Question

Theorem 7 implies that for n large enough, L^{n+1} is k -ordered hamiltonian if $L^n(G)$ is k -ordered hamiltonian. Can we remove the dependence on the size of the minimum degree, as is the case in Theorem 1?

Question 12. *If G is k -ordered hamiltonian, is $L(G)$ k -ordered hamiltonian?*

When G is 4-ordered hamiltonian, $L(G)$ is 4-ordered hamiltonian. This can be shown by assigning to each ordered edge in G one of its endvertices; as in the proof of Theorem 7, these vertices are ordered in a hamiltonian cycle. The cycle is then expanded to a complete closed list of adjacent edges that contains the four ordered edges in the correct order; this list becomes a hamiltonian cycle in $L(G)$.

When G is 5-ordered hamiltonian, $L(G)$ is 5-ordered hamiltonian. The proof is similar to that for 4-ordered hamiltonian graphs, but a few small modifications must be made similar to those in Figure 1 because there may not be enough endvertices for a one-to-one assignment of endvertices to edges. For $k > 5$, this method no longer works because there are too many possible choices of edges where no one-to-one assignment of endvertices to ordered edges can be made.

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