On a question of Sós about 3-uniform friendship hypergraphs

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Abstract

The well-known Friendship Theorem states that if G is a graph in which every pair of vertices has exactly one common neighbor, then G has a single vertex joined to all others (a "universal friend"). This property uniquely defines the graph for a given number of vertices. V. Sós defined an analogous friendship property for 3-uniform hypergraphs, and gave a construction satisfying the friendship property that has a universal friend. She also asked whether any other 3-uniform hypergraphs satisfy the friendship property. We answer this question affirmatively by demonstrating hypergraphs without a universal friend that satisfy the friendship property. These hypergraphs were found using integer programming.

1 Introduction

The well-known Friendship Theorem states that if G is a graph in which every pair of vertices has a common neighbor, then G has a single vertex joined to all others. In fact, such a graph G exists only for odd values of n, and when it exists it is unique: G consists of $(n-1)/2$ triangles joined at a single vertex. This graph has become known as the "friendship graph." The earliest published proof of this theorem is due to Erdős et al. [4]. Since then, a variety of different proofs have appeared, see Huneke [5].

Many generalizations of this theorem have been studied. One approach has been to consider graphs in which every set of k vertices has exactly d common neighbors; in [8] and

[2], it was shown that the only such graph is the complete graph on $k+d$ vertices. In [3], this generalization is extended to infinite graphs. Perhaps the most studied generalization has been the search for graphs in which any two vertices are connected by a unique path of length k, known as p_k-graphs. A survey of this idea can be found in [1].

In [7], Sós presented an entirely different generalization of the friendship problem. She proposed studying 3-uniform hypergraphs with the following property.

The Friendship Property for 3-uniform Hypergraphs: For every three vertices x, y, z, there exists a unique vertex w such that xyw, yzw, and xzw are all edges in the hypergraph.

Sós observed that for some values of n, there is a 3-uniform hypergraph satisfying the Friendship Property that (as in the original graph version) features a "universal friend" that is in an edge with every other vertex.

Proposition 1. *(Sós [7]) When $n \equiv 2 \mod 6$ or $n \equiv 4 \mod 6$, there exists a hypergraph $\mathcal{H}$ satisfying the friendship property such that some vertex w of $\mathcal{H}$ appears in an edge with every pair of vertices x, y.*

Proof. We construct $\mathcal{H}$ with the stated properties. The edge set of $\mathcal{H}$ must contain all $\binom{n-1}{2}$ 3-sets containing the vertex w. Since $n \equiv 2, 4 \mod 6$, there is a Steiner triple system on the $n-1$ vertices of $\mathcal{H}$ remaining when w is removed. (See [9] for information on Steiner triple systems.) Add an edge to $\mathcal{H}$ for each set in the Steiner triple system.

We verify that $\mathcal{H}$ satisfies the Friendship Property. Let $\mathcal{V}$ be the vertex set of $\mathcal{H}$. For any three vertices x, y, z with $x, y, z \neq w$, the edges xyw, yzw, and xzw all appear in $\mathcal{H}$. Further, since the edges avoiding x were derived from a Steiner triple system, w is the only vertex with this property. (For any $u \in \mathcal{V} - \{w, x, y, z\}$, uxy and uxz cannot both be edges in the hypergraph.) For any three vertices w, x, y, the pair x, y is in exactly one edge that does not contain w, say xyz. Now xyz, wyz, and wxz are all edges in $\mathcal{H}$, and again the friendship property is satisfied. □

Sós asked whether other 3-uniform friendship hypergraphs exist. In this paper, we report the results of using Integer Programming (IP) techniques to locate these constructions. Our main results are as follows: For $n \leq 10$, $n \neq 8$, the only 3-uniform friendship hypergraphs are those found by Sós. However, for $n = 8$, $n = 16$, and $n = 32$, there are hypergraphs satisfying the Friendship Property that are not isomorphic to the Sós construction.

Throughout this paper, unless stated otherwise, all hypergraphs are 3-uniform. The vertex set and the edge set of a hypergraph $\mathcal{H}$ will be denoted $\mathcal{V}(\mathcal{H})$ and $\mathcal{E}(\mathcal{H})$, respectively. Edges in a hypergraph will be denoted as xyz. A hypergraph is a *friendship hypergraph* if it satisfies the Friendship Property. Any friendship hypergraph with a universal friend as described in Proposition 1 is a *universal friend hypergraph*; any other friendship hypergraph is a *non-universal friend hypergraph*. If $\mathcal{H}$ is a friendship hypergraph, then w is the *completion* of x, y, z if w is the unique vertex satisfying $wxy, wxz, wyz \in \mathcal{E}(\mathcal{H})$. We us K_4^3 to denote a complete 3-uniform hypergraph on four vertices.

2 Elementary Observations

We begin with some elementary properties of friendship hypergraphs.

Observation. *(a) Every pair of vertices appears in at least one edge together.*

(b) Every edge must be contained in a unique K_4^3.

Proof. (a) Let $x, y \in \mathcal{V}(\mathcal{H})$ and $z \neq x, y$. Then the triple x, y, z has some completion w; the edge xyw is in $\mathcal{H}$, and hence x and y appear together in an edge.

(b) Let $xyz \in \mathcal{E}(\mathcal{H})$. The triple x, y, z has a unique completion w, hence xyw, yzw, and xzw are also in $\mathcal{E}(\mathcal{H})$. Therefore the vertices x, y, z, w induce a K_4^3. Uniqueness follows from the uniqueness of w.

□

Observation (b) implies that the edges of a friendship hypergraph partition into K_4^3's. We will focus our attention on this partition, since knowing the K_4^3 structure tells us everything about the edge structure. We will refer to the number of K_4^3's containing a vertex x as the K_4^3-degree of x; clearly observation (b) implies that the degree of a vertex is 4 times the K_4^3-degree.

The simple observations above produce rough bounds on the number of K_4^3's in a friendship hypergraph. By (b), every set of three vertices is in at most one K_4^3, so the number of K_4^3's is at most $\binom{n}{3}/4$. By (a) and (b) together, every pair of vertices is in some K_4^3. Since there are 6 distinct pairs of vertices covered per K_4^3, the number of K_4^3's is at least $\binom{n}{2}/6$. We can improve this lower bound slightly.

Proposition 2. *If $\mathcal{H}$ is a friendship hypergraph, then there are at least $\binom{n-1}{2} + \frac{1}{2}n - 1 = \frac{1}{2}n(n-2)$ edges in $\mathcal{H}$.*

Proof. Remove some vertex a from $\mathcal{H}$, and consider the hypergraph $\mathcal{H}'$ that remains containing edges both of size 2 and size 3. Let E_2 be the edges of size 2, E_3 the edges of size 3, $E_3^A \subset E_3$ the edges contained in a K_4^3 with a, and $E_3 \setminus E_3^A = E_3^B$. For any pair of vertices x, y, there must be exactly one w that is the completion for x, y, a in $\mathcal{H}$. Hence there must be exactly one w such that $xw \in E_2$, $yw \in E_2$, and $xyw \in E_3$. Since there are $\binom{n-1}{2}$ such pairs in $\mathcal{V}(\mathcal{H})$, this accounts for $\binom{n-1}{2}$ edges of size 3; however, some edges have been counted more than once. We consider edges in E_3^A and E_3^B separately.

Any triple in E_3^A has been counted 3 times, since $xyw \in E_3^A$ implies that $xy, yw, xw \in E_2$, and thus xyw is counted for all three pairs xy, yw and xw. Thus $|E_3^A| = \frac{1}{3}|E_2|$.

On the other hand, if $xy \notin E_2$, then x cannot be the completion for y, w, a and y cannot be the completion for x, w, a, hence each such $xyw \in E_3^B$ has been counted only once, and $|E_3^B| \geq \binom{n-1}{2} - |E_2|$. Note that we have inequality, since there may be some edges of E_3^B that are not used to create completions for triples containing a.

The number of edges in $\mathcal{H}$ is $|E_3^A| + |E_3^B| + |E_2|$, so $\mathcal{H}$ has at least $\binom{n-1}{2} - |E_2| + \frac{1}{3}|E_2| + |E_2| = \binom{n-1}{2} + \frac{1}{3}|E_2|$ edges.

We seek a lower bound on $|E_2|$. We claim that $G = \mathcal{H}'$ restricted to E_2 forms a graph with diameter at most 2 such that every edge is contained in at least one triangle. For any pair of vertices $x, y \in \mathcal{V}(\mathcal{H})$, there is some completion w for the triple x, y, a, hence $x \to w \to y$ is a path of length 2 connecting x to y. If xy is an edge, then x, y, w form a triangle, thus every edge of G is on a triangle. We next claim that G has at least $\frac{3}{2}n - 3$ edges. Let T be a spanning tree of G. Since G has $n-1$ vertices, T has $n-2$ edges. Each of these edges must lie in a triangle. Since any additional edge completes a triangle with at most 2 edges of T, G must have have at least $(n-2)/2$ additional edges, and G has at least $\frac{3}{2}n - 3$ edges. Combined with the result above, we conclude there are at least $\binom{n-1}{2} + \frac{1}{2}n - 1 = \frac{1}{2}n(n-2)$ edges in $\mathcal{H}$. $\square$

Dividing this lower bound by 4 yields a lower bound on the number of K_4^3's.

Corollary 3. *$\mathcal{H}$ contains at least $\left\lceil \frac{n(n-2)}{8} \right\rceil$ K_4^3's.*

Note that a universal friend hypergraph, if it exists, has $\binom{n-1}{2}/3 = \frac{1}{6}(n-1)(n-2)$ K_4^3's.

The argument in the proof of Proposition 2 holds for every vertex x, so choosing x to maximize $|E_2|$ gives the best possible lower bound using this method. We also observe that the total number of edges in $\mathcal{H}$ is always *exactly* $\binom{n}{2} + \frac{1}{3}|E_2| + A(x)$, where $A(x)$ is the number of edges that are not used to complete any triples containing the vertex x and E_2 is defined as above. A better understanding of the quantity $A(x)$ may lead to improvements in the bound on the number of edges possible.

3 The Integer Program

We next present an explanation of the integer program we used to search for friendship hypergraphs. We represent our vertex set of $\mathcal{H}$ as $\{0, \ldots, n-1\}$. The program consists of two types of variables: x and y. To each 4-subset A of $\mathcal{V}(\mathcal{H})$, we assign a binary variable x_A that indicates the presence of a K_4^3 on the vertex set A. To each set $S \subseteq \mathcal{V}(\mathcal{H})$ of size 3 and each vertex $v \notin S$, we assign a binary variable y_S^v that indicates whether S is a nonedge *and* v is the completion for S.

To ensure that every 3-set S of vertices has exactly one completion, we include the constraint

$$\sum_{A \supset S} x_A + \sum_{v \notin S} y_S^v = 1.$$

If S is an edge in the hypergraph, it is contained in exactly one K_4^3, hence $\sum_{A \supset S} x_A = 1$ if S is an edge, and 0 otherwise. If S is not contained in a K_4^3 (and thus is a nonedge), then its completion is some unique v, and hence $\sum_{v \notin S} y_S^v = 1$.

In order to ensure that a feasible solution can actually be realized as a hypergraph, we need to ensure that if v is the completion for $S = \{s_1, s_2, s_3\}$, then each of vs_1s_2, vs_2s_3, and vs_1s_3 must be in some K_4^3. For a fixed $S = \{s_1, s_2, s_3\}$ and $v \notin S$, let $B_1 = \sum_{w \notin S, w \neq v} x_{\{v,s_1,s_2,w\}}$, let $B_2 = \sum_{w \notin S, w \neq v} x_{\{v,s_2,s_3,w\}}$, and let $B_3 = \sum_{w \notin S, w \neq v} x_{\{v,s_1,s_3,w\}}$. Note that B_1 indicates whether vs_1s_2 is an edge of the hypergraph, and similarly for B_2 and B_3. We need $y_S^v = 1$ if and only if each B_i is 1. To achieve this with linear constraints, for each S and for each v we added the constraints

$$y_S^v \geq B_1 + B_2 + B_3 - 2$$

$$y_S^v \leq B_1$$

$$y_S^v \leq B_2$$

$$y_S^v \leq B_3.$$

For values for which a universal friend hypergraph exists, we searched for an alternate friendship hypergraph by forcing the maximum degree to be less than $\binom{n-1}{2}$. We solved this IP using CPLEX [6], in all cases testing for feasibility rather than optimality in some parameter. The amount of symmetry in the problem slowed the solver down significantly. To combat this, we fixed 012 as a nonedge and vertex 3 as its completion.

Based on the computer search, we have the following result.

Theorem 4. *There does not exist any friendship hypergraph for* $n = 5, 6, 7, 9$*. Furthermore, for* $n = 10$*, the only friendship hypergraphs are universal friend hypergraphs.*

A more refined search yielded the following result.

Theorem 5. *Up to isomorphism, there are exactly two friendship hypergraphs on 8 vertices: the universal friend hypergraph and the hypergraph* $\mathcal{F}^8$ *consisting of* K_4^3*'s on the vertex sets* $\{0123\}$, $\{0145\}$, $\{0167\}$, $\{2345\}$, $\{2367\}$, $\{4567\}$, $\{0246\}$, *and* $\{1357\}$.

Proof. We leave it to the reader to verify the friendship property in $\mathcal{F}^8$. Notice that the universal friend hypergraph has 7 K_4^3's and maximum K_4^3-degree 7, while $\mathcal{H}$ has 8 K_4^3's with each vertex having K_4^3-degree of 4.

In order to establish uniqueness, we ran the IP with the following additional constraints. First we set the IP to maximize the number of K_4^3's, and the result was 8. Next we restricted the K_4^3-degree of each vertex to be at most 6, thus prohibiting the universal friend hypergraph, and minimized the number of K_4^3's in a friendship hypergraph; the result was again 8. Therefore all non-universal friend constructions must have exactly eight K_4^3's.

We next enforced exactly eight K_4^3's in the IP, and maximized the K_4^3-degree of vertex 0. The result was 4, and since the total K_4^3-degree was $8 \cdot 4 = 32$, any such friendship hypergraph must be 4-regular. We next considered the number of K_4^3's containing a particular pair of vertices, which we call pair degree. In $\mathcal{F}^8$, for every vertex i there is some other vertex j with which i appears in three K_4^3's. No pair of vertices can be in all four K_4^3's together, otherwise some edge would be contained in more than one K_4^3. Hence the pair degree for a 4-regular friendship hypergraph is at most 3. To rule out any possible friendship hypergraphs with some vertex in at most two K_4^3's with any other vertex, we added a constraint requiring vertex 0 to be in at most two K_4^3's with any other vertex. With this addition, the IP was infeasible. Therefore all possible non-universal friend constructions have every vertex in four K_4^3's, and each vertex has another vertex with which it shares three K_4^3's.

Finally, it remained to show that $\mathcal{F}^8$ is the only friendship hypergraph with these properties. We enforced pair degree 3 for vertices 0 and 1, 2 and 3, 4 and 5, and 6 and 7, and fixed $x_{0,1,2,3} = 0$; that is, we forced two of the fixed pairs not to be in a K_4^3 together. With this restriction, the IP was infeasible, hence all of the fixed pairs appear in a K_4^3 together

Table 1: A Friendship Hypergraph with 8 vertices

	4 structures	Pair Partition?	Additional K_4^3
$\mathcal{F}^8$	K_4^3 on vertices 0 through 3; K_4^3 on vertices 4 through 7	All K_4^3's of the form $\{2i, 2i+1, 2j, 2j+1\}$, $0 \leq i \leq 1$, $2 \leq j \leq 3$	0,2,4,6 1,3,5,7
No. of K_4^3	2	4	2
			Total number of K_4^3: 8

as in $\mathcal{F}^8$. Finally, we notice that once we have fixed those six K_4^3's, there are only two K_4^3's remaining. As every vertex must be in one more K_4^3, the final K_4^3's must be $\{0246\}$ and $\{1357\}$ (up to isomorphism). □

4 Larger Non-Universal Friend Constructions

Using the IP, we were also able to establish the following.

Theorem 6. *There exist friendship hypergraphs without a universal friend for* $n = 16$ *and* $n = 32$*. Furthermore, for* $n = 16$*, there exist at least three nonisomorphic constructions.*

Descriptions of the constructions $\mathcal{F}^8, \mathcal{F}_1^{16}, \mathcal{F}_2^{16}, \mathcal{F}_3^{16}$, and $\mathcal{F}^{32}$ can be found in tables 1, 2, and 3. As the IP was too large to solve for these large values of n without additional constraints, we observe some important properties of the constructions that enabled us to add new constraints to the IP. Let $\mathbb{F} = \{\mathcal{F}_1^{16}, \mathcal{F}_2^{16}, \mathcal{F}_3^{16}, \mathcal{F}^{32}\}$.

First, notice that each $\mathcal{F}^n \in \mathbb{F}$ contains two disjoint copies of a friendship hypergraph on $n/2$ vertices. We call this the *inductive property.* Also, with the exception of $\mathcal{F}_3^{16}$, the vertices of each $\mathcal{F}^n \in \mathbb{F}$ partition into pairs such that each pair appears in a K_4^3 with each other pair. We refer to this as a *pair partition property.* Enforcing these two properties enabled us to find the construction $\mathcal{F}_1^{16}$.

Perhaps the most interesting property satisfied by all hypergraphs in $\mathbb{F}$ is that they have nontrivial automorphisms. If we view a vertex v of $\mathcal{F}^n \in \mathbb{F}$ as an element of $(\mathbb{Z}_2)^{\log n}$ and we allow $a \in (\mathbb{Z}_2)^{\log n}$ to act on v by $v \mapsto v + a$, then we see that $\mathcal{F}^n$ is fixed under this action. In other words, if we view the vertices as binary $(\log n)$-tuples, then any map that flips some fixed subset of the $\log n$ bits is an automorphism of $\mathcal{F}^n$. We call this the *automorphism property.* Note that this property also implies that each $\mathcal{F}^n \in \mathbb{F}$ is regular and vertex-transitive.

Table 2: Friendship Hypergraphs with 16 vertices

	8 structures	Pair Partition?	Additional K_4^3
$\mathcal{F}_1^{16}$	$\mathcal{F}^8$ on vertices 0 through 7; $\mathcal{F}^8$ on vertices 8 through 15	All K_4^3's of the form $\{2i, 2i+1, 2j, 2j+1\}$, $0 \le i \le 3$, $4 \le j \le 7$	0,2,8,10 1,5,9,13 3,7,8,12 0,2,13,15 1,5,10,14 3,7,11,15 0,4,8,12 1,6,9,14 4,6,9,11 0,4,11,15 2,5,10,13 4,6,12,14 0,7,8,15 2,6,9,13 5,7,8,10 1,3,9,11 2,6,10,14 5,7,13,15 1,3,12,14 3,4,11,12
No. of K_4^3	16	16	20
			Total number of K_4^3: 52
$\mathcal{F}_2^{16}$	$\mathcal{F}^8$ on vertices 0 through 7; $\mathcal{F}^8$ on vertices 8 through 15	All K_4^3's of the form $\{2i, 2i+1, 2j, 2j+1\}$, $0 \le i \le 3$, $4 \le j \le 7$	0,3,9,10 1,4,10,15 3,4,8,15 0,3,12,15 1,5,8,12 3,5,11,13 0,4,9,13 1,6,10,13 3,6,8,13 0,5,11,14 1,7,9,15 3,7,10,14 0,6,8,14 2,4,10,12 4,7,8,11 0,7,11,12 2,5,9,14 4,7,13,14 1,2,8,11 2,6,11,15 5,6,9,10 1,2,13,14 2,7,9,12 5,6,12,15
No. of K_4^3	16	16	24
			Total number of K_4^3: 56
$\mathcal{F}_3^{16}$	$\mathcal{F}^8$ on vertices 0 through 7; $\mathcal{F}^8$ on vertices 8 through 15	none	0,1,14,15 1,4,9,12 3,4,9,14 0,2,12,14 1,5,10,14 3,5,10,12 0,2,13,15 1,5,11,15 3,5,11,13 0,3,8,11 1,6,10,13 3,6,10,15 0,3,9,10 1,6,11,12 3,6,11,14 0,4,10,14 1,7,8,14 3,7,8,12 0,4,11,15 1,7,9,15 3,7,9,13 0,5,8,13 2,3,12,13 4,5,10,11 0,5,9,12 2,4,10,12 4,6,8,10 0,6,8,14 2,4,11,13 4,6,9,11 0,6,9,15 2,5,8,15 4,7,12,15 0,7,10,13 2,5,9,14 4,7,13,14 0,7,11,12 2,6,8,12 5,6,12,15 1,2,8,11 2,6,9,13 5,6,13,14 1,2,9,10 2,7,10,15 5,7,8,10 1,3,12,14 2,7,11,14 5,7,9,11 1,3,13,15 3,4,8,15 6,7,8,9 1,4,8,13
No. of K_4^3	16	0	52
			Total number of K_4^3: 68

Table 3: A Friendship Hypergraph with 32 vertices

	16 structures	Pair Partition?	Additional K_4^3 containing 0 (†)
$\mathcal{F}^{32}$	$\mathcal{F}_2^{16}$ on vertices 0 through 15; $\mathcal{F}_2^{16}$ on vertices 16 through 31	All K_4^3's of the form $\{2i, 2i+1, 2j, 2j+1\}$, $0 \leq i \leq 7$, $8 \leq j \leq 15$	0,3,17,18 0,6,25,31 0,11,22,29 0,3,28,31 0,6,26,28 0,12,19,31 0,4,17,21 0,8,16,24 0,12,22,26 0,4,18,22 0,8,21,29 0,13,21,24 0,4,24,28 0,9,22,31 0,14,22,24 0,4,27,31 0,10,17,27 0,15,17,30 0,6,17,23 0,11,19,24 0,15,20,27 0,6,18,20
No. of K_4^3	104	64	22 containing 0, 176 total
			Total number of K_4^3: 344

(†) Remaining K_4^3's can be found using vertex transitivity of the contruction.

$\mathcal{F}_2^{16}$ was found by enforcing the inductive property, the pair partition property, and the automorphism property. In order to find $\mathcal{F}_3^{16}$, we dropped the constraints enforcing the pair partition property; hence this is the only construction in $\mathbb{F}$ that does not satisfy it. Finally, $\mathcal{F}^{32}$ was found by again enforcing all three properties. Table 3 shows the result when we fixed two disjoint copies of $\mathcal{F}_2^{16}$. When instead two copies of $\mathcal{F}_1^{16}$ were fixed, the solver found a construction isomorphic to $\mathcal{F}^{32}$.

The automorphism property has an interesting consequence. Recall the following facts from group theory:

(a) If a group G acts on a set X, then X is the disjoint union of the orbits of the action, where the orbit of $x \in X$ is $\{gx : g \in G\}$.

(b) If a finite group G acts on a set X, then the number of elements in any orbit is a divisor of the order of G.

Proposition 7. *If $\mathcal{F}^n$ is a friendship hypergraph on n vertices satisfying the automorphism property, then the number of K_4^3's contained in $\mathcal{F}^n$ is a multiple of $n/4$.*

Proof. Let $K = \{K_4^3 \in \mathcal{F}^n\}$. View $(\mathbb{Z}_2)^{\log n}$ as a group under addition. Notice that the automorphism property implies that $(\mathbb{Z}_2)^{\log n}$ acts on K via addition, that is $K \ni \{a, b, c, d\} \mapsto \{a + x, b + x, c + x, d + x\}$ for all $x \in (\mathbb{Z}_2)^{\log n}$ is a group action. Let S be an orbit of this action, and let $A = \{a, b, c, d\}$ be an element of S. Note that every element x of the vertex set must appear in some K_4^3 in S, by considering the action of $x + a$ on A. Thus, $\cup_{B \in S} B$ is the entire vertex set, and since each element of K has size 4, $|S| \geq n/4$. Since the order of the group is 2^n, fact (a) implies that $|S|$ is a multiple of $n/4$. Fact (b) thus implies that $|K|$ is the sum of integers divisible by $n/4$, and hence $|K|$ is divisible by $n/4$. □

5 Conclusion

The discovery of these constructions leads us to the following conjecture.

Conjecture 8. *For all k, there exists a friendship hypergraph on 2^k vertices satisfying the inductive property, the pair property, and the automorphism property.*

Furthermore, based on the results of our computer search for small values of n, we conjecture the following:

Conjecture 9. *If n is odd, then there is no friendship hypergraph on n vertices.*

There are clearly many additional questions to be answered in this area. Are there other values of n for which non-universal friend constructions exist? In the original question posed for graphs, the first step in each proof of the uniqueness of the friendship graph is to prove that any construction without a universal friend must be regular. Is there a corresponding property that must hold for 3-uniform hypergraphs?

The current lower bound on the number of edges in a friendship hypergraph is quadratic, whereas the upper bound is cubic. Is the number of edges in a universal friend construction (for values of n for which they exist) a lower bound on the number of edges in any friendship hypergraph on the same number of vertices? Can the upper bound be improved?

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