The Elimination Procedure for the Competition Number is Not Optimal

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Abstract

Given an acyclic digraph D, the competition graph $C(D)$ is defined to be the undirected graph with $V(D)$ as its vertex set and where vertices x and y are adjacent if there exists another vertex z such that the arcs (x,z) and (y,z) are both present in D. The competition number $k(G)$ for an undirected graph G is the least number r such that there exists an acyclic digraph F on $|V(G)|+r$ vertices where $C(F)$ is G along with r isolated vertices. Kim and Roberts [3] introduced an elimination procedure for the competition number, and asked whether the procedure calculated the competition number for all graphs. We answer this question in the negative by demonstrating a graph where the elimination procedure does not calculate the competition number. This graph also provides a negative answer to a similar question about the related elimination procedure for the phylogeny number found in [2].
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1 Introduction

Given an acyclic digraph D, the competition graph $C(D)$ is defined to be the undirected graph with $V(D)$ as its vertex set and where vertices x and y are adjacent if there exists another vertex z such that the arcs (x,z) and (y,z) are both present in D. From the ecological origins of competition graphs, z is known as a *prey* of x and y. Competition graphs were

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introduced by Cohen [1] and have been widely studied since then. For a survey on competition graphs and the related phylogeny graphs, see Roberts [6].

In [7] Roberts noted that for any graph G, G along with r isolated vertices is the competition graph of some acyclic digraph if r is sufficiently large. The competition number $k(G)$ is defined to be the least such r. Roberts considered using an elimination procedure to calculate $k(G)$. An elimination procedure takes as input G and an ordering $P = v_1, \ldots, v_n$ of the vertices of G and produces an acyclic digraph F such that $C(F) = G \cup I_r$; that is, the competition graph of F is G along with r isolated vertices. The procedure "eliminates" each vertex in order by ensuring that all of the edges incident on the vertex will appear in $C(F)$. The goal is to create an elimination procedure that outputs an acyclic digraph F where $|V(F) \setminus V(G)| = k(G)$. Opsut [4] found an example of a graph G where Roberts' original elimination procedure does not calculate the competition number $k(G)$, thus giving a counterexample to Roberts' conjecture that the procedure always calculates $k(G)$. Kim and Roberts [3] then modified the elimination procedure and asked whether their modified procedure works for all graphs. They were able to show that the modified version calculates the competition number for a large class of graphs, the so-called "kite-free" graphs. In this paper, we present a graph L where Kim and Roberts' elimination procedure does not always calculate the competition number, in the following sense: for each order P of vertices of L, the elimination procedure can produce an acyclic digraph with more than $k(L)$ additional vertices.

This graph also is a counterexample to a similar question about the related elimination procedure for the phylogeny number. Given an acyclic digraph D, the phylogeny graph $P(D)$ is defined to be the undirected graph with $V(D)$ as its vertex set and with adjacencies as follows: two vertices x and y are adjacent if one of the arcs (x, y) or (y, x) is present in D, or if there exists another vertex z such that the arcs (x, z) and (y, z) are both present in D. Phylogeny graphs were introduced by Roberts and Sheng [8] from an idealized model for reconstructing phylogenetic trees in molecular biology. For a simple graph G, the phylogeny number $p(G)$ is the least number r such that there exists an acyclic digraph D on $|V(G)|+r$ vertices where G is an induced subgraph of $P(D)$. Hartke [2] introduced an elimination procedure for the phylogeny number based on Kim and Roberts' modified elimination procedure for the competition number. Hartke also asked whether this procedure calculated the phylogeny number for all graphs. The graph L shown in this paper also shows that the elimination procedure for the phylogeny number does not calculate the phylogeny number for all graphs.

Elimination procedures which seek to determine a graph-theoretical parameter through step-wise elimination of vertices have various applications in graph theory and its applications. A classic example is a perfect elimination sequence which can be used to determine if a graph is triangulated. Roberts [6] was led to consider an elimination procedure for the competition number through variants of perfect elimination used by Parter [5] and Rose [9] in connection with numerical analysis.

Note that the focus of creating elimination procedures is not on efficiency, since calculating the competition number or the phylogeny number with an elimination procedure requires $n!$ runs (one for each ordering of the vertices). In fact, calculating both the competition number ([4]) and the phylogeny number ([8]) have been shown to be NP-complete. Instead, the focus is on whether an elimination procedure *could* be created that exactly calculates the relevant number. Our result shows that this might be much more difficult than originally thought.

1.1 The Elimination Procedure

We present here the elimination procedure of Kim and Roberts as described in [3]. The input graph G that we wish to calculate the competition number of need not be connected. We use $N_G[v]$ to denote the closed neighborhood of v in G: the set of vertices in G that are adjacent to v, along with v itself.

The Elimination Procedure for the Competition Number

Input: A graph $G = (V, E)$, and an ordering $P = v_1, v_2, \ldots, v_n$ of the vertices of G.

Output: An acyclic digraph F_n on V together with M new vertices such that $C(F_n)$ is $G \cup I_M$.

Initialization: Set F_0 to the digraph with vertex set V and no arcs.
Set $G_0 := G$. G_j is a spanning subgraph of G that contains the edges of G that do not appear in $C(F_{j-1})$.
Set $H_0 := G$.
Set $S_0 := \emptyset$. S_j is a set of vertices available as prey.

j^{th} Iteration, $j = 0, \ldots, n-1$: Let $\mathcal{E}_j = \{K_{j_1}, \ldots, K_{j_{h_j}}\}$ be a minimum size vertex clique covering of $N_{G_j}[v_{j+1}]$ by maximal cliques of H_j, ordered arbitrarily. (Note that $v_{j+1} \in K_{j_s}$ for $s = 1, \ldots, h_j$.) Let $h_j = |\mathcal{E}_j|$. Form G_{j+1} from G_j by removing the edges of K_i from G_j for all i.
Form the digraph F_{j+1} by adding vertices and arcs to F_j as follows: Pick h_j distinct vertices $u_1, \ldots, u_{j_{h_j}}$ from S_j. If $|S_j| < h_j$, then add $h_j - |S_i|$ additional vertices $u_{j_{h_j} - |S_i|}, \ldots, u_{j_{h_j}}$ to F_{j+1}. For each clique $C_i \in \mathcal{E}_j$, add the arcs (w, u_i) to F_{j+1} for each $w \in C_i$.
Form S_{j+1} by $S_{j+1} := \Big(S_j \setminus \{u_1, \ldots, u_{j_{h_j}}\}\Big) \cup \{v_{j+1}\}$. Obtain H_{j+1} from H_j by deleting the vertex v_{j+1} and all incident edges.

The *elimination number* $M(G, P, \mathcal{E})$ of a graph G and an ordering P is the number of vertices added to F_n so that $C(F_n)$ is $G \cup I_{M(G,P,\mathcal{E})}$. Here the notation reflects both the explicit parameters G and P, as well as the choices of clique covers $\mathcal{E}_j$ made during each iteration of the procedure. The elimination number $M(G)$ of just the graph G is the minimum of $M(G, P, \mathcal{E})$ taken over all orders P of the vertices, but with an arbitrary choice of clique covers $\mathcal{E}$ for each order. Thus, $M(G)$ may not be well-defined. Lemma 24 of [2] shows that there is always a "right" clique cover for each order such that the minimum attains the competition number $k(G)$. However, the elimination procedure provides no way to choose this right clique cover. Thus, in demonstrating that the elimination procedure is not optimal for some graph G, we need to show that for each order P, there is a choice $\mathcal{E}$ of clique covers so that $M(G, P, \mathcal{E}) > k(G)$.

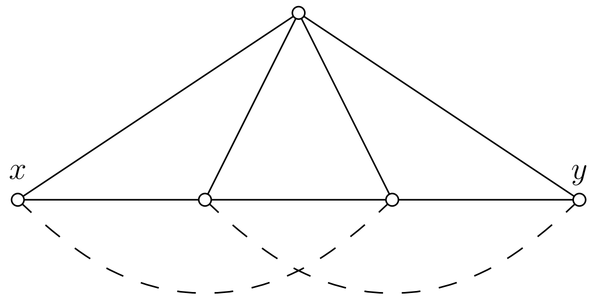

Figure 1: A kite. The solid edges must be present, the dotted edges cannot be present, and the edge xy may or may not be present.

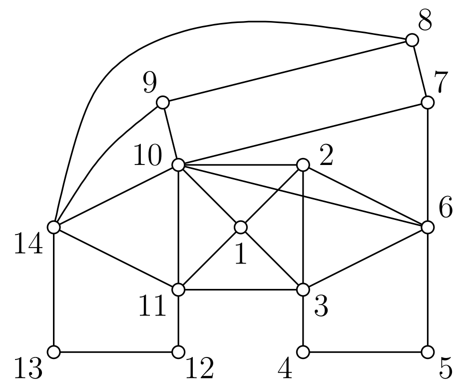

Figure 2: The graph L.

1.2 The Counterexample

Kim and Roberts [3] proved that their elimination procedure is optimal for kite-free graphs, where a kite is the configuration found in Figure 1. An alternate proof of this theorem can be found in [2]. From this theorem, any graph where the elimination procedure is not optimal must contain a kite. The graph L in Figure 2 contains two kites on the vertices $\{1, 2, 3, 10, 11\}$ and $\{1, 2, 3, 6, 10\}$. When eliminating vertices 1 or 2 first, two different clique covers of two triangles each can be used to eliminate the incident edges. One of these choices is a *good* choice for the edge clique cover, but one is a *bad* choice. Our effort in constructing the counterexample is to force 1 or 2 to be eliminated first, so that a bad choice is made. When 1 or 2 is not eliminated first, then we will show that no choices allow the elimination procedure to attain the competition number.

Proposition 1. *For each ordering P of the vertices of L, there is a choice of edge clique coverings $\mathcal{E}$ such that $M(L, P, \mathcal{E}) > 2$.*

Proof. Let $P = v_1, v_2, \ldots, v_{14}$ be an ordering of the vertices of L. We consider several cases:

Case 1. $v_1 = 1$.

We make the bad choice of the cliques $\{1, 2, 3\}$ and $\{1, 10, 11\}$. Any choice other than vertex 2 can not be eliminated without increasing the number of extra vertices added to F. But after 2 is eliminated, no vertex has its remaining incident edges coverable by a single clique. Thus, $M(L, P, \mathcal{E}) > 2$.

Case 2. $v_1 = 2$.
We make the bad choice of the cliques $\{2, 1, 3\}$ and $\{2, 6, 10\}$. Vertex 1 is the only vertex that can then be eliminated without increasing the number of added vertices, but then no vertex has its remaining incident edges coverable by a single clique.

Case 3. $v_1 =$ 3, 6, 10, 11, or 14.
Each of these vertices requires at least three cliques to cover its incident edges.

Case 4. $v_1 =$ 4 or 5.
One of these vertices can be eliminated using two cliques, and the other is then the only vertex that can be eliminated without increasing the number of added vertices. But then no vertex has its remaining incident edges coverable by a single clique.

Case 5. $v_1 = 7$.
Vertex 7 can be eliminated with two cliques, and then 8 and 9 are the only vertices that can then be eliminated without increasing the number of added vertices. But then no vertex has its remaining incident edges coverable by a single clique.

Case 6. $v_1 =$ 8 or 9.
One of these vertices can be eliminated using two cliques, and then the other vertex and 7 are the only vertices that can then be eliminated without increasing the number of added vertices. But then no vertex has its remaining incident edges coverable by a single clique.

Case 7. $v_1 =$ 12 or 13.
One of these vertices can be eliminated using two cliques, and the other is the only vertex that can then be eliminated without increasing the number of added vertices. But then no vertex has its remaining incident edges coverable by a single clique.

Thus, there exists a choice $\mathcal{E}$ of clique cover such that $M(L, P, \mathcal{E}) > 2$ for any order P. □

Proposition 2. *The competition number of* L *is* 2.

Proof. First note that there is no vertex in L whose incident edges can be covered with one clique. Thus, $k(L) \geq 2$. But the elimination procedure using the order $1, 2, \ldots, 14$ and the good choice of cliques $\{1, 2, 10\}$ and $\{1, 3, 11\}$ for vertex 1 produces an elimination number $M(L, P, \mathcal{E})$ of 2. Thus, $k(L) = 2$. □

2 The Elimination Procedure for the Phylogeny Number is Not Optimal

Because of the similarities in the elimination procedures for the competition and phylogeny numbers, the same graph L is also a counterexample to the optimality of the elimination procedure for the phylogeny number. Both of the following propositions are proved in a fashion similar to the propositions above.

Proposition 3. *For each ordering P of the vertices of L, there is a choice of edge clique coverings $\mathcal{E}_i$ such that the number of added vertices is greater than* 1*.*

Proposition 4. *The phylogeny number of L is* 1*.*

3 Conclusion

Given the existence of the graph L where Kim and Roberts' elimination procedure is not optimal, the main question is whether a different elimination procedure can be created that will calculate the competition number for all graphs. However, Kim and Roberts' procedure is still of interest, particularly in determining when it is optimal and when it is not. For instance, is L the smallest graph where the elimination procedure fails, or is there a smaller counterexample? Is there a counterexample with only one kite? Kites without the xy edge do not always admit a choice in clique covers. Is the elimination procedure optimal when there is no choice? A complete characterization of when the procedure is optimal and when it is not is still open.

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