Oral Qualifying Exam Syllabus

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Committee: Profs. J. Beck, F. Roberts, E. Boros

Graph Theory and Combinatorics

Graph Theory

Basics of graphs.

Königsberg bridge problem, Eulerian circuits, application to De Bruijn cycles.

Spanning trees, Prim's and Kruskal's algorithms, connection to matroids, Matrix Tree Theorem.

Planarity, Euler's formula, Kuratowski's theorem.

Digraphs, tournaments, strong connectivity, orientability, Robbins' theorem, arc-vulnerability, vertex-vulnerability, transitive orientability, Ghouila-Houri theorem.

Adjacency and incidence matrices, eigenvalues, Harary's theorem.

Vertex and edge connectivity, block decomposition, Menger's theorem, line graphs and edge Menger's theorem, König's theorem, Hall's theorem, matchings and coverings, 1-factors, Tutte's theorem, Dilworth's theorem, flows in graphs, max flow-min cut, integrality.

Intersection graphs, Marczewski's theorem, interval graphs, circular arc graphs, line graphs, clique graphs, food webs, competition graphs.

Forbidden subgraph problems, Mantel's theorem, counting triangles in G and $\overline{G}$, Moser-Moon theorem, Turán's theorem, Erdős-Stone theorem, Erdős-Simonovits theorem.

Hamiltonian cycles and paths, Dirac's theorem, Ore's theorem, cycle lengths in 2-connected graphs, Erdős-Gallai theorem, Bondy's theorem on pancyclic graphs, Erdős-T. Sós conjecture.

Coloring problems: chromatic number, Brook's theorem, edge chromatic number, Vizing's theorem, bipartite case, Szekeres-Wilf theorem, five color theorem, four color theorem, chromatic number not a local property, perfect graphs, perfect graph theorem, construction of triangle free graphs with arbitrarily high chromatic number, list coloring and H-coloring.

Subdivisions and minors, topological subgraphs, Hajos' conjecture, the topological clique number and chromatic number of a.a. graphs, Hadwiger's conjecture.

References:

R. Diestel, *Graph Theory*, Graduate Texts in Mathematics no. 173, 2000.

D.B. West, *Introduction to Graph Theory*, Prentice Hall, 1996.

F. Roberts, *Discrete Mathematical Models with Applications to Social, Biological, and Environmental Problems*, Prentice Hall, 1976.

Combinatorics

Basics of counting, generating functions, binomial coefficients, recursive sequences, Stirling's formula.

Sperner's theorem, LYM inequality, Littlewood-Offord problem.

Intersecting families, Erdős-Ko-Rado theorem.

Frankl and Wilson's Ramsey construction, Hadwiger's problem on coloring the plane and Frankl's theorem, Borsuk conjecture and Kahn-Kalai counterexample.

2 coloring hypergraphs: uniform hypergraphs, bounded degree hypergraphs, almost disjoint hypergraphs, Erdős-Selfridge theorem, Beck-Fiala theorem.

Ramsey's theorem, König's lemma, compactness arguments, Van der Waerden's theorem, Hales-Jewett theorem, Ramsey numbers, stepping up lemma, general upper bounds, probabilistic lower bounds, Ajtai-Komlós-Szemerédi upper bound for $R(3, n)$, matching lower bound.

Reference:

J. van Lint and R. Wilson, *A Course in Combinatorics*. Cambridge University Press, 1992.

The Probabilistic Method

Basics of probability, linearity of expectation, inclusion-exclusion, Bonferroni inequalities, Binomial and Poisson distributions are approximately normal, central limit theorem and law of large numbers, conditional probabilities, law of total probability.

Random graphs, monotone properties, threshold functions, relationship between being connected and having no isolated vertices.

The second moment method, threshold function for having a certain graph as a subgraph.

The local lemma, hypergraph discrepancy, Ramsey numbers, application to coloring problems.

Reference:

N. Alon and J. Spencer, *The Probabilistic Method*, Wiley, 2000.

Discrete Optimization

Linear programming duality theorem, Farkas' lemma, linear and integer programs, the parameters ν, ν_κ, ν^*, τ, τ_κ, and τ^* of a hypergraph and their l.p. significance.

Minimax theorems: Menger's, König's, max-flow min-cut, Hoffman-Gale circulation theorem, Dilworth's; proofs via Hoffman-Kruskal theorem.

Perfect graphs and associated hypergraphs, perfect graph theorem and conjecture, properties of minimal non-perfect graphs, classes of perfect graphs, Lovász' θ function, Shannon capacity of a graph, connection to semidefinite programming.

Tutte's 1-factor theorem, Edmonds' matching polyhedron, Gallai structure theorem, use in Lovász' perfect matching algorithm, edge-coloring and fractional edge-coloring, Chinese postman problem, T-joins and T-cuts, Lucchesi-Younger theorem.

Four color problem, nowhere zero flows.

Total unimodularity, Ghouila-Houri theorem, other characterizations.

Approximability of hard problems: stability, max-cut, chromatic number; polynomial algorithms for max-cut on planar graphs and weakly bipartite graphs.

Stable marriage problem, algorithms, integrality of polyhedra.

References:

L. Lovász and M.D. Plummer, *Matching Theory*, North-Holland, 1986.

W.J. Cook, W.H. Cunningham, W.R. Pulleyblank, and A. Schrijver, *Combinatorial Optimization*, J. Wiley, 1998.

A. Schrijver, *Theory of Linear and Integer Programming*, J. Wiley, 1986.