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Derivation of the Black-Scholes Equation

Rating

Mathematically Mature: may contain mathematics beyond calculus with proofs.

Question of the Day

What is the most important idea in the derivation of the binomial option pricing model?

Key Concepts

1. The derivation of the Black-Scholes equation uses
   (a) tools from calculus,
   (b) the quadratic variation of Geometric Brownian Motion,
   (c) the no-arbitrage condition to evaluate growth of non-risky portfolios,
   (d) and a simple but profound insight to eliminate the randomness or risk.

Vocabulary

1. A **backward parabolic PDE** is a partial differential equation of the form $V_t + DV_{xx} + \dots = 0$ with highest derivative terms in t of order 1 and highest derivative terms x of order 2 respectively. **Terminal values** $V(S, T)$ at an end time $t = T$ must be satisfied in contrast to the initial values at $t = 0$ required by many problems in physics and engineering.

2. A **terminal condition** for a backward parabolic equation is the specification of a function at the end time of the interval of consideration to uniquely determine the solution. It is analogous to an initial condition for an ordinary differential equation, except that it occurs at the end of the time interval, instead of the beginning.

Mathematical Ideas

Explicit Assumptions Made for Modeling and Derivation

For mathematical modeling of a market for a risky security we will ideally assume

1. that a large number of identical, rational traders always have complete information about all assets they are trading,
2. changes in prices are given by a continuous random variable with some probability distribution,
3. that trading transactions take negligible time,
4. purchases and sales can be made in any amounts, that is, the stock and bond are divisible, we can buy them in any amounts including negative amounts (which are short positions),
5. the risky security issues no dividends.

The first assumption is the essence of what economists call the **efficient market hypothesis**. The efficient market hypothesis leads to the second assumption as a conclusion, called the **random walk hypothesis**. Another version of the random walk hypothesis says that traders cannot predict the direction of the market or the magnitude of the change in a stock so the best predictor of the market value of a stock is the current price. We will make the second assumption stronger and more precise by specifying the probability distribution of the changes with a stochastic differential equation. The remaining hypotheses are simplifying assumptions which can be relaxed at the expense of more difficult mathematical modeling.

We wish to find the value V of a derivative instrument based on an underlying security which has value S. Mathematically, we assume

1. the price of the underlying security follows the stochastic differential equation
$$dS = rS\,dt + \sigma S\,dW$$
or equivalently that $S(t)$ is a Geometric Brownian Motion with parameters $r - \sigma^2/2$ and σ,

2. the risk free interest rate r and the volatility σ are constants,

3. the value V of the derivative depends only on the current value of the underlying security S and the time t, so we can write $V(S, t)$,

4. All variables are real-valued, and all functions are sufficiently smooth to justify necessary calculus operations.

The first assumption is a mathematical translation of a strong form of the efficient market hypothesis from economics. It is a reasonable modeling assumption but finer analysis strongly suggests that security prices have a higher probability of large price swings than Geometric Brownian Motion predicts. Therefore the first assumption is not supported by data. However, it is useful since we have good analytic understanding of Geometric Brownian Motion.

The second assumption is a reasonable assumption for a modeling attempt although good evidence indicates neither interest rates nor the volatility are constant. On reasonably short times scales, say a period of three months for the lifetime of most options, the interest rate and the volatility are approximately constant. The third and fourth assumptions are mathematical translations of the assumptions that securities can be bought and sold in any amount and that trading times are negligible, so that standard tools of mathematical analysis can be applied. Both assumptions are reasonable for modern security trading.

Derivation of the Black-Scholes equation

We consider a simple derivative instrument, an option written on an underlying asset, say a stock that trades in the market at price $S(t)$. A payoff function $\Lambda(S)$ determines the value of the option at expiration time T. For $t < T$, the option value should depend on the underlying price S and the time t. We denote the price as $V(S, t)$. So far all we know is the value at the final time $V(S, T) = \Lambda(S)$. We would like to know the value $V(S, 0)$ so that we know an appropriate buying or selling price of the option.

As time passes, the value of the option changes, both because the expiration date approaches and because the stock price changes. We assume the option price changes smoothly in both the security price and the time. Across a short time interval δt we can write by the Taylor series expansion of V that:

$$\delta V = V_t \delta t + V_s \delta S + \frac{1}{2} V_{SS} (\delta S)^2 + \dots$$

The neglected terms are of order $(\delta t)^2$, $\delta S \delta t$, and $(\delta S)^3$ and higher. We rely on our intuition from random walks and Brownian motion to explain why we keep the terms of order $(\delta S)^2$ but neglect the other terms. More about this later.

To determine the price, we construct a **replicating portfolio**. This will be a specific investment strategy involving only the stock and a cash account that will yield exactly the same eventual payoff as the option in all possible future scenarios. Its present value must therefore be the same as the present value of the option and if we can determine one we can determine the other. We thus define a portfolio Π consisting of $\phi(t)$ shares of stock and $\psi(t)$ units of the cash account $B(t)$. The portfolio constantly changes in value as the security price changes randomly and the cash account accumulates interest.

In a short time interval, we can take the changes in the portfolio to be

$$\delta \Pi = \phi(t) \delta S + \psi(t) r B(t) \delta t$$

since $\delta B(t) \approx r B(t) \delta t$, where r is the interest rate. This says that short-time changes in the portfolio value are due only to changes in the security price, and the interest growth of the cash account, and not to additions or subtraction of the portfolio amounts. Any additions or subtractions are due to subsequent reallocations financed through the changes in the components themselves.

The difference in value between the two portfolios changes by

$$\delta(V - \Pi) = (V_t - \psi(t) r B(t))\delta t + (V_S - \phi(t))\delta S + \frac{1}{2} V_{SS} (\delta S)^2 + \dots.$$

This could be considered to be a three-part portfolio consisting of an option, and short-selling ϕ units of the security and ψ units of bonds.

Next come a couple of linked insights: As an initial insight we will eliminate the first order dependence on S by taking $\phi(t) = V_S$. Note that this means the rate of change of the derivative value with respect to the security value determines a policy for $\phi(t)$. Looking carefully, we see that this policy removes the "randomness" from the equation for the difference in values! (What looks like a little "trick" right here hides a world of probability theory. This is really a Radon-Nikodym derivative that defines a change of measure that transforms a diffusion, i.e. a transformed Brownian motion with drift, to a standard Wiener measure.)

Second, since the difference portfolio is now *non-risky*, it must grow in value at exactly the same rate as any risk-free bank account:

$$\delta(V - \Pi) = r(V - \Pi)\delta t.$$

This insight is actually our now familiar no-arbitrage-possibility argument: If $\delta(V - \Pi) > r(V - \Pi)\delta t$, then anyone could borrow money at rate r to acquire the portfolio $V - \Pi$, holding the portfolio for a time δt, and then selling the portfolio, with the growth in the difference portfolio more than enough to cover the interest costs on the loan. On the other hand if $\delta(V - \Pi) < r(V - \Pi)\delta t$, then short-sell the option in the marketplace for V, purchase $\phi(t)$ shares of stock and loan the rest of the money out at rate r. The interest growth of the money will more than cover the repayment of the difference portfolio. Either way, the existence of risk-free profits to be made in the market will drive the inequality to an equality.

So:

$$r(V - \Pi)\delta t = (V_t - \psi(t) r B(t))\delta t + \frac{1}{2} V_{SS} (\delta S)^2.$$

Recall the quadratic variation of Geometric Brownian Motion is deterministic, namely $(\delta S)^2 = \sigma^2 S(t)^2 \delta t$,

$$r(V - \Pi)\delta t = (V_t - \psi(t) r B(t))\delta t + \frac{1}{2}\sigma^2 S^2 V_{SS} \delta t.$$

Cancel the δt terms, and recall that $V - \Pi = V - \phi(t)S - \psi(t)B(t)$, and $\phi(t) = V_S$, so that on the left $r(V - \Pi) = rV - rV_S S - r\psi(t)B(t)$. The terms $-r\psi(t)B(t)$ on left and right cancel, and we are left with the Black-Scholes equation:

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0.$$

Note that under the assumptions made for the purposes of the modeling the partial differential equation depends only on the constant volatility σ and the constant risk-free interest rate r. This partial differential equation (PDE) must be satisfied by the value of any derivative security depending on the asset S.

Some comments about the PDE:

- The PDE is linear: Since the solution of the PDE is the worth of the option, then two options are worth twice as much as one option, and a portfolio consisting two different options has value equal to the sum of the individual options.

- The PDE is **backwards parabolic** because of the form $V_t + (1/2)\sigma^2 S^2 V_{SS}$. Thus, **terminal values** $V(S, T)$ (in contrast to the initial values required by many problems in physics and engineering) must be specified. The value of a European option at expiration is known as a function of the security price S, so we have a terminal value. The PDE is solved to determine the value of the option at times before the expiration date.

Comment on the derivation:

The derivation above follows reasonably closely the original derivation of Black, Scholes and Merton. Option prices can also be calculated and the Black-Scholes equation derived by more advanced probabilistic methods. In this equivalent formulation, the discounted price process $\exp(-rt)S(t)$ is shifted into a "risk-free" measure using the Cameron-Martin-Girsanov Theorem, so that it becomes a martingale. The option price $V(S, t)$ is then the discounted expected value of the payoff $\Lambda(t)$ in this measure, and the PDE is obtained as the backward evolution equation for the expectation. The derivation above follows the classical derivation of Black and Scholes, but the probabilistic view is more modern and can be more easily extended to general market models.

The derivation of the Black-Scholes equation above uses the fairly intuitive partial derivative equation approach because of the simplicity of the derivation. This derivation:

- is easily motivated and related to similar derivations of partial differential equations in physics and engineering,
- avoids the burden of developing additional probability theory and measure theory machinery, including filtrations, sigma-fields, previsibility, and changes of measure including Radon-Nikodym derivatives and the Cameron-Martin-Girsanov theorem.
- also avoids, or at least hides, martingale theory that we have not yet developed or exploited,
- does depend on the stochastic process knowledge that we have gained already, but not more than that knowledge!

The disadvantages are that:

- we are forced to skim certain details relying on motivation instead of strict mathematical rigor,
- when we are done we still have to solve the partial differential equation to get the price on the derivative, whereas the probabilistic methods deliver the solution almost automatically and give the partial differential equation as an auxiliary by-product,
- the probabilistic view is more modern and can be more easily extended to general market models.

Sources

This derivation of the Black-Scholes equation is drawn from "Financial Derivatives and Partial Differential Equations" by Robert Almgren, in *American Mathematical Monthly*, Volume 109, January, 2002, pages 1–11.

Problems to Work for Understanding

1. Show by substitution that two exact solutions of the Black-Scholes equations are

   (a) $V(S,t) = AS$, A some constant.

   (b) $V(S,t) = A exp(rt)$

   Explain in financial terms what each of these solutions represents. That is, describe a simple "claim", "derivative" or "option" for which this solution to the Black Scholes equation gives the value of the claim at any time.

2. Draw the expiry diagrams (that is, a graph of terminal condition of portfolio value versus security price S) at the expiration time for the portfolio which is

   (a) Short one share, long two calls with exercise price K. (This is called a **straddle** .)

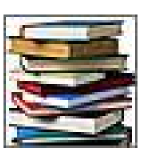
   (b) Long one call, and one put both exercise price K. (This is also called a straddle.)

   (c) Long one call, and two puts, all with exercise price K. (This is called a **strip** .)

   (d) Long one put, and two calls, all with exercise price K. (This is called a **strap** .)

   (e) Long one call with exercise price K_1 and one put with exercise price K_2. Compare the three cases when $K_1 > K_2$, (known as a **strangle**), $K_1 = K_2$, and $K_1 < K_2$.

   (f) As before, but also short one call and one put with exercise price K. (When $K_1 < K < K_2$, this is called a **butterfly spread.**)



Reading Suggestion:

References

[1] R. Almgren. Financial derivatives and partial differential equations. *The American Mathematical Monthly*, 109:1–12, 2002.

Outside Readings and Links:

1. Bradley University, School of Business Administration, Finance Department, Kevin Rubash. A very brief description on the development history of option theory and the Black-Scholes model for calculating option value, with the notations, Greeks and some explanatory graphs. Also contains a calculators for the option value calculation. Submitted by Yogesh Makkar, November 19, 2003.

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